SELECTED PROBLEMS

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1. Transformations of Euclidean Space and Clifford Geometric Algebra

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1.1 Similarity transformations

Any direct similarity transformation of Euclidean affine space $\mathcal{E}^n$ is a product of an orientation-preserving homothety and a proper rigid motion. In other words any direct similarity transformation of $\mathcal{E}^n$ preserves the orientation and the angles (see [1] for details). Let $Sim^+(\mathcal{E}^n)$ denote the group of all direct similarities of $\mathcal{E}^n$.

If we identify the Euclidean plane $\mathcal{E}^2$ with the field of complex numbers $\mathbb{C}$, then $f \in Sim^+(\mathcal{E}^2)$ has a representation

$$f(\mathbf{z}) = \alpha\,\mathbf{z} + \beta\ , \qquad \mathbf{z} \in \mathbb{C} \cong \mathcal{E}^2,$$

where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in \mathbb{C}$ are constant.

In the case $n = 3$, we may identify three-dimensional Euclidean space $\mathcal{E}^3$ with the space of pure quaternions $\mathrm{Im}\,\mathbb{H}$. Then an arbitrary similarity $f \in Sim^+(\mathcal{E}^3)$ has a representation

$$f(\mathbf{z}\ ) = \lambda\ \mathbf{n}\ \mathbf{z}\ \mathbf{n}^{-1}\ +\ \mathbf{a}, \qquad \mathbf{z}\ \in \mathrm{Im}\,\mathbb{H} \cong \mathcal{E}^3,$$

where $\lambda \in \mathbb{R}^+$, $\mathbf{n}$ is a fixed unit quaternion and $\mathbf{a}$ a fixed pure quaternion. Clearly, f is a homothety whenever $\lambda \neq 1$ and $\mathbf{n} = \pm 1$, f is a translation whenever $\lambda = 1$ and $\mathbf{n} = \pm 1$, f is a rotation whenever $\lambda = 1$ and $\mathbf{a} = 0$.

The next step is the discovery of a common representation of the similarities by one equation in any dimension.

Problem 1. Using Clifford geometric algebra find a representation of $f \in Sim^+(\mathcal{E}^n)$ for any $n > 1$ in terms of the geometric product.

Note that the dilations, rotations and translations can be represent by rotors (see [3]).

1.2 Quadratic transformations

Let Q be a non-degenerate quadric hypersurface in the real projective space $\mathbb{P}^n$, $n > 1$, let p be a point lying on Q, and let τ be the tangent hyperplane to Q at p. Then the pair $(Q,\ p)$ determines a quadratic involutory transformation

$$\varphi \ : \ \mathbb{P}^n \backslash \tau \quad \longrightarrow \mathbb{P}^n \backslash \tau$$

with the set of fixed points $Q\backslash\ p$. In fact for $x \in \mathbb{P}^n\backslash\tau$,

$$\varphi(x) = x' = \mathrm{Pol}_Q(x) \cap \langle px \rangle \, ,$$

where $\mathrm{Pol}_Q(x)$ is the polar hyperplane of x and $\langle px \rangle$ is the straight line passing through p and x. If $\mathbb{P}^n$ is the projective extension of $\mathcal{E}^n$ and τ is the hyperplane at infinity, then φ is a one-to-one transformation of $\mathcal{E}^n$. The three-dimensional Clifford algebra $\mathcal{G}_3$ and techniques from [4] and [5] are applied in [2] for a description of the case $n = 2$.

If $\quad Q = p + \lambda\, q + \lambda^2\, r \quad (\lambda \in \mathbb{R}) \quad$ is a conic in $\mathbb{P}^2$, where $p \in Q$, q and r are non-collinear points in the projective plane, the line $L = p \wedge x = \langle px \rangle$ passes through p and x, and meets the conic Q at a second point x^0 (see [2]).

Then φ has the equation

$$x' = 2p \cdot x\, \left(x^0 \cdot x - x \cdot x \right)\, p + \left(p \cdot x\ p \cdot x^0 - p \cdot p\ x^0 \cdot x \right) x,$$

where the second intersection point $x^0 = [rpx]^{\,2} p - [qpx]\ [rpx]\, q + [qpx]^{\,2} r$, $[\,p\, q\, r\,] = (\,p \wedge q \wedge r\,) . I^{-1}$, and I is the unit pseudoscalar in $\mathcal{G}_3$.

Problem 2. Using Clifford geometric algebra obtain an explicit equation for φ for $n \geq 3$.

The techniques stated in [6] can be used for the solution of the above problems.

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2. On the Distribution of Kloosterman Sums on Polynomials over Quaternions

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Abstract

We formulate two problems concerning the distribution of angles of Kloosterman sums and the algebraic structure of curves over quaternions.

Keywords: density of a distribution, elliptic curve, Kloosterman sum, quaternions.

2.1 The Problem of the distribution of Kloosterman sums

Let

$$T_p(c,d) = \sum_{x=1}^{p-1} e^{2\pi i (\frac{cx+\frac{d}{x}}{p})}$$

$$1 \le c, d \le p-1;\ x, c, d \in \mathbf{F}_p^*$$

be the Kloosterman sum.
By the A. Weil [1] estimate

$$T_p(c,d) = 2\sqrt{p}\cos\theta_p(c,d),$$

$$\theta_p(c,d) = \arccos(\frac{T_p(c,d)}{2\sqrt{p}})$$

on the interval $[0, \pi)$:

Problem 1. *Let c and d be fixed and p vary over all primes not dividing c and d. What is the distribution of angles $\theta_p(c,d)$ as $p \to \infty$?*

Motivation. Kloosterman sums of different types have in recent years played an increasingly important role in the study of many problems in analytic number theory [2].
During 1983 and 1989 the presenter of this problem implemented computations with Kloosterman sums $T_p(c,d)$ for primes $p,\ 2 \le p \le 13499,\ c = d = 1$, and for $1 \le c \le p-1,\ 1 \le d \le p-1$, for some primes p [3, 4].
Conjecture. *Under the conditions of Problem 1 the angles $\theta_p(c,d)$ are distributed on $[0, \pi)$ with Sato-Tate density $\frac{2}{\pi}\sin^2 t$.*

2.2 On algebraic structure of curves over quaternions

Let E be the elliptic curve over a field K, $E(K)$ the group of rational points of E. $E(k)$ is an abelian group with a finite number of generators (Poincaré, Mordell, Weil).
Let $End(E) = Hom(E,E)$ be the ring of endomorphisms of E. In some cases the ring $End(E)$ is noncommutative (Deuring).
Example. Let $K = \mathbf{F}_{2^2}$ and E be defined by the equation

$$y^2 + y = x^3.$$

Let $\alpha(x,y) = (\varepsilon x, y)$ where $\varepsilon^2 + \varepsilon + 1,\ \varepsilon \in \mathbf{F}_{2^2}$ and Frobenius $\phi(x,y) = (x^2, y^2)$. Then, since $\varepsilon^2 \neq \varepsilon,\ \alpha \circ \phi(x,y) \neq \phi \circ \alpha(x,y)$. Moreover the ring $End(E)$ is isomorphic to the set of integer quaternions $End(E) = \{m_1 + m_2 i + m_3 j + m_4 k,\ m_i \in \mathbf{Z}$ or $m_i + 1/2 \in \mathbf{Z}\}$, so E is a supersingular elliptic curve.

Let $\mathbf{H} = \{a_1 + a_2 i + a_3 j + a_4 k,\ a_i \in \mathbf{R}\}$ be the ring of quaternions and

$$EH : y^2 = x^3 + ax + b$$

be the equation with $a, b \in \mathbf{H}$.

Problem 2. *What can be said about the algebraic properties and structure of the set*

$$EH(\mathbf{H}) = \{(x,y) \in \mathbf{H} \times \mathbf{H}\ \&\ y^2 = x^3 + ax + b\}$$

and the set $Mor(EH(\mathbf{H}))$ of maps $EH(\mathbf{H}) \to EH(\mathbf{H})$?

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3. Harmonic Sliding Analysis Problems

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Harmonic sliding analysis (HSA) is a dynamic spectrum analysis [1] in which the next analysis interval differs from the previous one by including the next signal sample and excluding the first one from the previous analysis interval. Such a harmonic analysis is necessary for time-frequency localization [2] of the analysed signal given peculiarities. Using the well-known Fast Fourier transform (FFT) is not effective in this context. More effective are known recursive algorithms which use only one complex multiplication for computing one harmonic during each analysis interval. The presenter of this problem improved one of those algorithms, so that it became possible to use one complex multiplication for computing two, four and even eight (for complex signals) harmonics simultaneously [3], [4], [5]. One problem of HSA was mentioned in [6]. In [7] there is a short review of the presenter's papers devoted to HSA. A basic problem of HSA is: are there methods to further increase the speed of the response using up-to-date nanotechnology? If this is possible, how does one do it?

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4. Spectral Analysis under Conditions of Uncertainty

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In the classic spectral analysis, a sample $\{x_n, t_n,\ n = \overline{0, N-1}\}$ of the discrete time periodic process (x_n is the value of the process, t_n is the argument; N is the sample length) is expanded in the spectrum $C = \{C_k = (\sum_{n=0}^{N-1} x_n \bar{\phi}_k(t_n))/N,\ k = \overline{0, N-1}\}$ by using some basis system $\{\phi_k(t_n),\ k = \overline{0, N-1};\ n = \overline{1, N-1}\}$. The reverse representation of the process is $\{\widetilde{x}_n(C) = \sum_{k=0}^{N-1} C_k \phi_k(t_n)\}$, respectively.

Usually the sample is composed of the disturbed values with some model, for example, the model of disturbance can have the form $x_n = x_n^* + \varepsilon_n$, $n = \overline{0, N-1}$, where x_n^* is the unknown true value of the process, $\vec{\varepsilon} = \{\varepsilon_n\}$ is the additive disturbance.

If the disturbance has a stochastic nature and is described by some probability characteristics, there are classical approaches for description and computation of the spectrum for this stochastic process.

But in practice, often the statistical properties of disturbance are unknown, the researcher can show only the geometric constraint $|\varepsilon_n| \leq \varepsilon_{max}$, and the sample gives the collection of the *uncertainty sets* $\{H_n = [x_n - \varepsilon_{max}, x_n + \varepsilon_{max}]\}$. Additionally, the type of the process can be given, i.e., $x^*(t) = f(t, \vec{P})$, where $f(\cdot)$ is the describing function, $\vec{P}$ is the vector of parameters of dimension M, and $M < N$.

Now, each component C_k mentioned above becomes a set $\mathcal{C}_k = C_k + (\sum_{n=0}^{N-1} \varepsilon_n \bar{\phi}_k(t_n))/N$, where all $\varepsilon_n \in [-\varepsilon_{max}, \varepsilon_{max}]$, and the process spectrum $\mathcal{C}$ becomes the body in the space generated by the taken basis $\{\phi_k\}$.

The researcher is interested only in the vectors $C \in \mathcal{C}$ consistent with the shown collection of the uncertainty sets, i.e., $\widetilde{x}_n(C \in \mathcal{C}) \in H_n$ for all $n = \overline{0, N-1}$.

Problem. *Since the sets $\mathcal{C}_k$ depend on the vector of the disturbance $\vec{\varepsilon}$, they are not mutually independent and the spectrum $\mathcal{C}$ is not a rectangular parallellotop. How to describe and to compute the frontier of the body-spectrum $\mathcal{C}$ consistent with the collection of the uncertainty sets and the given type of the descriptive function?*

5. A Canonical Basis for Maximal Tori of the Reductive Centralizer of a Nilpotent Element

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Abstract We describe a problem that we partially solved in *[1], [2] and [3]* using a computational scheme. However a more conceptual argument might be very enlightening

5.1 Problem Description.

Let $\mathfrak{g}$ be a real semisimple Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Finally, let θ be the corresponding Cartan involution of $\mathfrak{g}$ and σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ the connected subgroup of the adjoint group $\mathfrak{G}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

A triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ is called a standard triple if $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$, e and $f \in \mathfrak{p}_{\mathbb{C}}$ then (x, e, f) is said to be normal. It is a result of Kostant and Rallis that any nilpotent e of $\mathfrak{p}_{\mathbb{C}}$ can be embedded in a standard normal triple (x, e, f). Let $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ be the centralizer of (x, e, f) in $\mathfrak{k}_{\mathbb{C}}$.

Maintaining the above notations we would like to solve the following problem:

5.2

PROBLEM: Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ such that $x \in \mathfrak{t}$. Then find two nilpotent elements e and f in $\mathfrak{p}_{\mathbb{C}}$ such that (x, e, f) is a standard triple and $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ is a maximal torus in $\mathfrak{k}_{\mathbb{C}}^{(x,ef)}$. More precisely give a natural basis for $\mathfrak{t}_1$.

The reader should be aware that in general $\mathfrak{t}_1 \neq \mathfrak{t}^{(x,e,f)}$ for an arbitrary e. A counterexample can be found in *[1]*. Furthermore, there is currently no good characterization of such a torus in the literature. Our conversation with several experts led us to believe that such a characterization may be quite technical. We wish to thank Prof. Andreas Dress from Bielefeld University for simplifying the statement of the problem.

5.3 Partial Results.

When $\mathfrak{g}$ is a real form of an exceptional non-compact simple complex Lie algebra we have developed a computational scheme to compute $\mathfrak{t}_1$. These results are found in *[1], [2], and [3]*. Here is a concise version of the algorithm:

Algorithm

Input: $\mathfrak{g}$ is a real exceptional simple Lie algebra, $\Delta_k = \{\beta_1, \ldots, \beta_l\}$ a set of fundamental roots of $\mathfrak{k}_{\mathbb{C}}$, and t is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ define by Δ_k.

Computation

- Compute x using the values of $\beta_i(x)$.
- Using x, express $e = \sum_{i=1}^{r} c_{\gamma_i} X_{\gamma_i}$, where γ_i is a non compact root. X_{γ_i} a non zero root vector and c_{γ_i} a complex number, in a regular semisimple subalgebra l_e of minimal rank r. Create the normal triple (x, e, f).
- Compute the intersection of the kernels of γ_i on t. *Observe that the complex span of such an intersection is a maximal torus in* $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$.

Output: $\mathfrak{t}_1$ is the complex span of the intersection computed above.

We do not have at this moment a good way of describing a natural basis for $\mathfrak{t}_1$ in general. The above choice of e suggests that e must be given as a linear combination of a minimal set of root vectors. In designing the above algorithm we used some information about the type of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ from the work of Dragomir Djokovič. We anticipate that we will need such information for the classical groups where the nilpotent orbits are parametrized by partitions.

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6. The Quantum Chaos Conjecture

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According to the general formulation of the quantum chaos conjecture introduced in [1] (see also [2]), the distribution of the distances between adjacent energy levels of a quantum system should be close to a Poisson distribution with density e^{-x} provided such a system is a quantum analogue of a classical integrable system. The quantum chaos conjecture has not been proved previously for any system. The quantum system is described by the Schrödinger equation and obtained from the classical system with the help of some standard procedures. We present a class of quantum systems containing as a special case the well-known and popular model of a rotating particle subject to δ-shocks ("kicked rotator"), which in many papers has been considered as one of the main objects in connection with confirming the conjecture.

The proof of the quantum chaos conjecture for this model makes essential use of results on the distribution of distances between adjacent fractional parts of the values of a polynomial, and the estimate of the remainder is based on a new theory of generalized continued fractions for number vectors ([3]-[6]). We consider a one-dimensional nonlinear oscillator determined by the Hamilton function $H = H(\phi, I, t) = H_0(I) + H_1(\phi, t)$:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial I} = \frac{\mathrm{d}H_0}{\mathrm{d}I}, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\partial H}{\partial \phi} = -\frac{\partial H_1}{\partial \phi}, \tag{1}$$

where I, ϕ are "action-angle" variables, t is an independent variable and the function $H_1(\phi, t)$ has period 2π with respect to ϕ and period

$T > 0$ with respect to t and can be represented as

$$H_1(\phi, t) = F(\phi) \sum_{k=-\infty}^{\infty} \delta(t - kT), \tag{2}$$

with $F(\phi)$ a smooth 2π-periodic function and $\delta = \delta(t)$ the delta-function.

We assume that $H_0(I) = \sum_{s=0}^{n} b_s I^s$ is a polynomial of degree $n \geq 2$ with coefficients $b_s = a_s/\hbar^s$ $(s = 0, \ldots, n)$, where $\hbar$ is the Planck constant and the a_s are real numbers. In the special case when $n = 2$, $a_0 = a_1 = 0$, and $F(\phi) = \gamma \cos\phi$ (γ is a constant), the system (1) represents a "kicked rotator". We introduce the Hilbert space L^2 of complex 2π-periodic functions as the state space of the quantum system and we define the momentum operator by $\hat{I} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$. The evolution of the wave function $\Psi = \Psi(\phi, t) \in L^2$ is described by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\phi, t) = \hat{H}(t)\Psi(\phi, t) \ ,$$

where $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$, $\hat{H}_0 = \sum_{s=0}^{n} b_s \hat{I}^s$, and $\hat{H}_1(t)$ is the limit as $\varepsilon \to 0$ $(\varepsilon > 0)$ of the operators of multiplication by the function $H_1^{(\varepsilon)}$ obtained from the function H_1 in (2) when the delta-function δ is replaced by a smooth positive function δ_ε supported on the interval $[0, \varepsilon]$ and having integral 1.

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7. Four Problems in Radar

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For more context on these four problems, please consult the chapter in this book by the same authors, from which this section is excerpted.

As analog hardware performance matures to a steady plateau, and Moore's Law provides for a predictable improvement in throughput and memory, it is only the advances in signal and data processing algorithms that offer potential for performance improvements in fielded sensor systems. However, it requires a revolution in system design and signal processing algorithms to dramatically alter the traditional architectures and concepts of operation. One important aspect of our current research emphasizes new and innovative sensors that are electrically small (on the order of 10 wavelengths or less), and operate in concert with a number of other electrically small sensor systems within a wide field of view (FOV). Our objective is to distribute the power and the aperture of the conventional wide area surveillance radar among a number of widely disbursed assets throughout the battlefield environment. Of course, we must have an algorithm for distributing those assets in real time as the dynamically changing surveillance demands. The mathematical challenge here relates to the traveling salesman problem. Classically, the traveling salesman must select his route judiciously in order to maximize potential sales. Recent analysis in the literature addresses multiples salesmen covering the same territory. This is analogous to our problem, where multiple unmanned aerial vehicle (UAV) based sensors are charged with the mission of detecting, tracking, and identifying all targets (friend or foe). Not only must these sensors detect and identify threat targets, they must also process data coherently across multiple platforms. Our mathematical challenge problem reduces to one in which the position and velocity all UAV-based sensors are selected to maximize detection performance and coverage area, and minimize revisit rate.

Enhancing one of the sensors described above, to be more like a classical radar with a large power-aperture product, leads to the second mathematical challenge problem to be addressed by this community. With a larger aperture and more precise estimates of target parameters (angle, Doppler), an opportunity to expand the hypothesis testing problem to include both detection and estimation emerges. Here, conventional wisdom dictates that we perform filtering and false alarm rate control as part of the detection process, yet perform track processing as a post-detection analysis, where the parameter estimation is focused upon target position and velocity history. Clearly, parameter estimation need not be accomplished as a post-detection process. Since this segmented approach to detection and track processing has been in effect for decades, it will require a dramatic demonstration of improvement before it will be embraced by the radar community.

A third challenge problem arises in the formulation of the Generalized Likelihood Ratio Test (GLRT). In Kelly's formulation of a GLRT, conditioning on finite sample support is incorporated into the basic test. As such, a statistical method developed under the assumption that only finite training data are available for sample covariance matrix formulation was made available. The next generalization to be made, in an extension of Kelly's GLRT, is to incorporate prior knowledge of the structure of the sample covariance matrix into the mathematical development of a statistical test. This mathematical structure arises due to the fact that the phase spectra of ground clutter as seen by an airborne radar is determined only by geometry, and remains independent of the underlying clutter statistics (except for initial phase). The effect of this geometric dependence is to localize the interference along a contour in the transform domain (Fourier analysis). Our objective is to formulate a single GLRT which incorporates the effects of finite training data as well as geometric dependence.

The fourth mathematical challenge facing the modern radar engineer is to incorporate adaptivity on transmit into the basic formulation of the signal processing algorithm. Since this is a new research topic, an opportunity exists to formulate the basic mathematical framework for fully adaptive radar on both transmit and receive. Further extensions arise by incorporating the above challenge problems into this analysis.