CLIFFORD GEOMETRIC ALGEBRAS IN MULTILINEAR ALGEBRA AND NON-EUCLIDEAN GEOMETRIES

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Abstract Given a quadratic form on a vector space, the geometric algebra of the corresponding pseudo-euclidean space is defined in terms of a simple set of rules which characterizes the geometric product of vectors. We develop geometric algebra in such a way that it augments, but remains fully compatible with, the more traditional tools of matrix algebra. Indeed, matrix multiplication arises naturally from the geometric multiplication of vectors by introducing a spectral basis of mutually annihiliating idempotents in the geometric algebra. With the help of a few more algebraic identities, and given the proper geometric interpretation, the geometric algebra can be applied to the study of affine, projective, conformal and other geometries. The advantage of geometric algebra is that it provides a single algebraic framework with a comprehensive, but flexible, geometric interpretation. For example, the affine plane of rays is obtained from the euclidean plane of points by adding a single anti-commuting vector to the underlying vector space. The key to the study of noneuclidean geometries is the definition of the operations of meet and join, in terms of which incidence relationships are expressed. The horosphere provides a homogeneous model of euclidean space, and is obtained by adding a second anti-commuting vector to the underlying vector space of the affine plane. Linear orthogonal transformations on the higher dimensional vector space correspond to conformal or Möbius transformations on the horosphere. The horosphere was first constructed by F.A. Wachter (1792–1817), but has only recently attracted attention by offering a host of new computational tools

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in projective and hyperbolic geometries when formulated in terms of geometric algebra.

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1. Geometric algebra

A Geometric algebra is generated by taking linear combinations of geometric products of vectors in a vector space taken together with a specified bilinear form. Here we shall study the geometric algebras of the *pseudo-euclidean* vector spaces $\mathcal{G}_{p,q} := \mathcal{G}_{p,q}(I\!R^{p,q})$ for which we have the indefinite metric

$$x \cdot y = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j$$

for $x = \begin{pmatrix} x_1 & \cdots & x_{p+q} \end{pmatrix}$ and $y = \begin{pmatrix} y_1 & \cdots & y_{p+q} \end{pmatrix}$ in $I\!R^{p,q}$. We first study the geometric algebra of the more familiar Euclidean space.

1.1 Geometric algebra of Euclidean n-space

We begin by introducing the geometric algebra $\mathcal{G}_n := \mathcal{G}(I\!R^n)$ of the familiar Euclidean n-space

$$I\!R^n = \{x |\ x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \text{ for } x_i \in I\!R\}.$$

Recall the dual interpretations of each element $x \in I\!R^n$, both as a point of $I\!R^n$ with the coordinates $\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and as the position vector or *directed line segment* from the origin to the point. We can thus express each vector $x \in I\!R^n$ as a linear combination of the *standard orthonormal basis vectors* $\{e_1, e_2, \cdots, e_n\}$ where $e_i = \begin{pmatrix} 0 & \cdots & 0 & 1_i & 0 & \cdots & 0 \end{pmatrix}$, namely

$$x = \sum_{i=1}^{n} x_i e_i.$$

The vectors of $I\!R^n$ are added and multiplied by scalars in the usual way, and the positive definite *inner product* of the vectors x and $y = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$ is given by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i. \tag{1}$$

The geometric algebra $\mathcal{G}_n$ is generated by the *geometric multiplication* and addition of vectors in $I\!R^n$. In order to efficiently introduce the geometric product of vectors, we note that the resulting geometric algebra $\mathcal{G}_n$ is isomorphic to an appropriate matrix algebra under addition and geometric multiplication. Thus, like matrix algebra, $\mathcal{G}_n$ is an associative, but non-commutative algebra, but unlike matrix algebra the elements of $\mathcal{G}_n$ are assigned a comprehensive geometric interpretation. The two fundamental rules governing geometric multiplication and its interpretation are:

- For each vector $x \in I\!R^n$,

$$x^2 = xx = |x|^2 = \sum_{i=1}^{n} x_i^2 \tag{2}$$

  where $|x|$ is the usual *Euclidean norm* of the vector x.

- If $a_1, a_2, \dots, a_k \in I\!R^n$ are k mutually orthogonal vectors, then the product

$$A_k = a_1 a_2 \dots a_k \tag{3}$$

  is totally antisymmetric and has the geometric interpretation of a *simple k-vector* or a *directed k-plane* .[1]

Let us explore some of the many consequences of these two basic rules. Applying the first rule (2) to the sum $a + b$ of the vectors $a, b \in I\!R^2$, we get

$$(a + b)^2 = a^2 + ab + ba + b^2,$$

or

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}(|a + b|^2 - |a|^2 - |b|^2)$$

which is a statement of the famous *law of cosines* . In the special case when the vectors a and b are orthogonal, and therefore anticommutative by the second rule (3), we have $ab = -ba$ and $a \cdot b = 0$.

If we multiply the orthonormal basis vectors $e_{12} := e_1 e_2$, we get the 2-vector or *bivector* e_{12}, pictured as the *directed plane segment* in Figure 1. Note that the *orientation* of the bivector e_{12} is counterclockwise, and that the bivector $e_{21} := e_2 e_1 = -e_1 e_2 = -e_{12}$ has the opposite or clockwise orientation.

[1] This means that the product changes its sign under the interchange of any two of the orthogonal vectors in its argument.

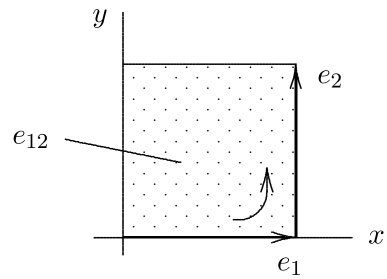

Figure 1. The directed plane segment $e_{12} = e_1e_2$.

We can now write down an orthonormal basis for the geometric algebra $\mathcal{G}_n$, generated by the orthonormal basis vectors $\{e_i|\ 1 \le i \le n\}$. In terms of the modified cartesian-like product, $\times_{i=1}^{n}(1, e_i) :=$

$$\{1, e_1, \dots, e_n, e_{12}, \dots, e_{(n-1)n}, \dots, \dots, e_{1\cdots(n-1)}, \dots, e_{2\cdots n}, e_{1\dots n}\}.$$

There are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

linearly independent elements in the standard orthonormal basis of $\mathcal{G}_n$. Any *multivector* or *geometric number* $g \in \mathcal{G}_n$ can be expressed as a sum of its homogeneous k-vector parts,

$$g = g_0 + \cdots + g_k + \cdots + g_n$$

where $g_k :=< g >_k = \sum_\sigma \alpha_\sigma e_\sigma$ where $\sigma = \sigma_1 \cdots \sigma_k$ for $1 \le \sigma_1 < \cdots < \sigma_k \le n$, and $\alpha_\sigma \in I\!R$. The real part $g_0 :=< g >_0 = \alpha_0 e_0 = \alpha_0$ of the geometric number g is just a real number, since $e_0 := 1$. By *definition*, any k-vector can be written as a linear combination of simple k-vectors or *k-blades* , [8, p.4].

Given two vectors $a, b \in I\!R^n$, we can decompose the vector a into components parallel and perpendicular to b, $a = a_\parallel + a_\perp$, where

$$a_\parallel = (a \cdot b)\frac{b}{|b|^2} = (a \cdot b)b^{-1},$$

and $a_\perp := a - a_\parallel$, see Figure 2.

With the help of (3), we now calculate the geometric product of the vectors a and b, getting

$$ab = (a_\parallel + a_\perp)b = a_\parallel \cdot b + a_\perp \wedge b = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \qquad (4)$$

$$a_{\|} = (a \cdot b)\frac{b}{|b|^2}$$
$$= (a \cdot b)b^{-1}.$$

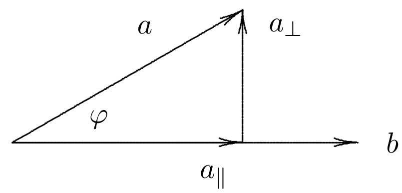

Figure 2. Decomposition of a into parallel and perpendicular parts.

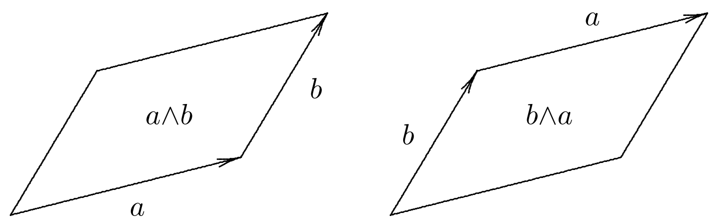

Figure 3. The bivectors $a \wedge b$ and $b \wedge a$.

where the *outer product* $a \wedge b := \frac{1}{2}(ab - ba) = a_\perp b = -ba_\perp = -b \wedge a$ is the bivector shown in Figure 3. The basic formula (4) shows that the geometric product ab is the sum of a scalar and a bivector part which characterizes the relative directions of a and b. If we make the assumption that a and b lie in the plane of the bivector e_{12}, then we can write

$$ab = |a||b|(\cos\varphi + I\sin\varphi) = |a||b|e^{I\varphi}, \tag{5}$$

where $I := e_{12} = e_1 e_2$ has the familiar property that

$$I^2 = e_1 e_2 e_1 e_2 = -e_1 e_2 e_2 e_1 = -e_1^2 e_2^2 = -1.$$

Equation (5) is the *Euler formula* for the geometric multiplication of vectors.

The definition of the inner product $a \cdot b$ and outer product $a \wedge b$ can be easily extended to $a \cdot B_r$ and $a \wedge B_r$, respectively, where $r \geq 0$ denotes the *grade* of the r-vector B_r:

Definition 1 *The inner product or contraction $a \cdot B_r$ of a vector a with an r-vector B_r is determined by*

$$a \cdot B_r = \frac{1}{2}(aB_r + (-1)^{r+1}B_r a) = (-1)^{r+1}B_r \cdot a.$$

Definition 2 *The outer product $a \wedge B_r$ of a vector a with an r-vector B_r is determined by*

$$a \wedge B_r = \frac{1}{2}(aB_r - (-1)^{r+1}B_r a) = -(-1)^{r+1}B_r \wedge a.$$

Note that $a \cdot \beta = \beta \cdot a = 0$ and $a \wedge \beta = \beta \wedge a = \beta a$ for the scalar $\beta \in I\!R$. Indeed, we will soon show that $a \cdot B_r =< aB_r >_{r-1}$ and $a \wedge B_r =< aB_r >_{r+1}$ for all $r \geq 1$; we have already seen that this is true when $r = 1$. There are different conventions regarding the use of the dot product and contraction [5, p. 35].

One of the most basic geometric algebras is the geometric algebra $\mathcal{G}_3$ of 3 dimensional Euclidean space which we live in. The complete standard orthonormal basis of this geometric algebra is

$$\mathcal{G}_3 = \times_{i=1}^{3}(1, e_i) = span\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}.$$

Any geometric number $g \in \mathcal{G}_3$ has the form $g = \alpha + v_1 + iv_2 + \beta i$ where $i := e_{123}$. Notice that we have expressed the bivector part of g as the *dual* of the vector v_2. Thus the geometric number $g = (\alpha + i\beta) + (v_1 + iv_2)$ can be expressed as the sum of its *complex scalar part* $(\alpha + i\beta)$ and a *complex vector part* $(v_1 + iv_2)$. Note that the complex scalar part has all the properties of an ordinary complex number $z = x + iy$. This follows easily from the fact that the pseudoscalar $i = e_{123}$ satisfies $i^2 = e_{123}e_{123} = e_{23}e_{23} = -1$.

We can use the Euler form (5) to see that

$$a = a(b^{-1}b) = (ab^{-1})b = \left(\frac{ab}{b^2}\right)b,$$

so the geometric quantity $ab/|b|^2$ *rotates and dilates* the vector b into the vector a *when multiplied by* b *on the left*. Similarly, multiplying b on right by $\frac{ba}{b^2}$ also rotates and dilates the vector b into the vector a. By re-expressing this result in terms of the Euler angle φ, letting $I = ie_3$, and assuming that $|a| = |b|$, we can write $a = \exp(ie_3\varphi)b = b\exp(-ie_3\varphi)$. Even more powerfully, and more generally, we can write

$$a = \exp(I\varphi/2)b\exp(-I\varphi/2),$$

which expresses the $\frac{1}{2}$-angle formula for rotating the vector $b \in I\!R^n$ in the plane of the simple bivector I through the angle φ. There are many more formulas for expressing reflexions and rotations in $I\!R^n$, or in the pseudo-euclidean spaces $I\!R^{p,q}$, [8], [10].

1.2 Basic algebraic identities

One of the most difficult aspects of learning geometric algebra is coming to terms with a host of unfamiliar algebraic identities. These important identities can be quickly mastered if they are established in a careful systematic way. The most important of these identities follows easily from the following two trivial algebraic identities involving the vectors a and b and an r-blade B_r where $r \geq 0$:

$$abB_r + bB_r a \equiv (ab + ba)B_r - b(aB_r - B_r a), \tag{6}$$

and

$$baB_r - aB_r b \equiv (ba + ab)B_r - a(bB_r + B_r b). \tag{7}$$

Whereas these identities are valid for general r-vectors, we state them here only for a simple r-vector B_r, the more general case following by linear superposition.

In proving the identities below, we use the fact that

$$bB_r =< bB_r >_{r-1} + < bB_r >_{r+1} . \tag{8}$$

This is easily seen to be true if B_r is an r-blade, in which case $B_r = b_1 \cdots b_r$ for r orthogonal, and therefore anticommuting, vectors $b_1, \ldots, b_r$. We then simply decompose the vector $b = b_\| + b_\perp$ into parts parallel and perpendicular to the subspace of $I\!R^n$ spanned by the vectors $b_1, \ldots, b_r$, and use the anticommutivity of the $b's$ to show that $b \cdot B_r = b_\| B_r = < b_\| B_r >_{r-1}$ and $b \wedge B_r = b_\perp B_r =< b_\perp B_r >_{r+1}$. This also shows the useful result that $b \cdot B_r$ and $b \wedge B_r$ are blades whenever B_r is a blade.

The following basic identity relates the inner and outer products:

$$a \cdot (b \wedge B_r) = (a \cdot b)B_r - b \wedge (a \cdot B_r), \tag{9}$$

for all $r \geq 0$. If $r = 0$, (9) follows from what has already been established. If $r \geq 2$ and even, (6) and definition (2) implies that

$$2a \cdot (bB_r) = 2(a \cdot b)B_r - 2b(a \cdot B_r).$$

Taking the r-vector part of this equation gives (9). If $r \geq 1$ and odd, (7) implies that

$$2b \cdot (aB_r) = 2(a \cdot b)B_r - 2a(b \cdot B_r)$$

which implies (9) by again taking the r-vector part of this equation and simplifying. By iterating (9), we get the important identity for *contraction*

$$a \cdot (b_1 \wedge \cdots \wedge b_n) = \sum_{i=1}^{n} (-1)^{i+1} (a \cdot b_i) \; b_1 \wedge \cdots \hat{i} \cdots \wedge b_n.$$

Let $I = e_{12\cdots n}$ be the unit pseudoscalar element of the geometric algebra $\mathcal{G}_{p,q} = \mathcal{G}(I\!R^{p,q})$. We give here a number of important identities relating the inner and outer products which will be used later in the contexts of projective geometry. For an r-blade A_r, the $(p+q-r)$-blade $A_r^* := A_r I^{-1}$ is called the *dual* of A_r in $\mathcal{G}_{p,q}$ with respect to the pseudoscalar I. Note that it follows that $I^* = II^{-1} = 1$ and $1^* = I^{-1}$. For $r+s \le p+q$, we have the important identity

$$(A_r \wedge B_s)^* = (A_r \wedge B_s)I^{-1} = A_r \cdot (B_s I^{-1}) = A_r \cdot B_s^* = (-1)^{s(p+q-s)} A_r^* \cdot B_s \tag{10}$$

1.3 Geometric algebras of psuedoeuclidean spaces

All of the algebraic identities discussed for the geometric algebra $\mathcal{G}_n$ hold in the geometric algebra with indefinite signature $\mathcal{G}_{p,q}$. However, some care must be taken with respect to the existence of non-zero *null vectors* . A non-zero vector $n \in I\!R^{p,q}$ is said to be a null vector if $n^2 = n \cdot n = 0$. The inverse of a non-null vector v is $v^{-1} = \frac{v}{v^2}$, so clearly a null vector has no inverse. The spacetime algebra $\mathcal{G}_{1,3}$ of $I\!R^{1,3}$, also called the *Dirac algebra*, has many applications in the study of the Lorentz transformations used in the special theory of relativity. Whereas non-zero null vectors do not exist in $I\!R^n$, there are many non-zero geometric numbers $g \in \mathcal{G}_n$ which are null. For example, let $g = e_1 + e_{12} \in \mathcal{G}_3$, then

$$g^2 = (e_1 + e_{12})(e_1 + e_{12}) = e_1^2 + e_{12}^2 = 1 - 1 = 0.$$

Let us consider in more detail the spacetime algebra $\mathcal{G}_{1,3}$ of $I\!R^{1,3}$. The standard orthonormal basis of $I\!R^{1,3}$ are the vectors $\{e_1, e_2, e_3, e_4\}$, where $e_1^2 = e_2^2 = e_3^2 = -1 = -e_4^2$. The standard basis of the bivectors $\mathcal{G}_{1,3}^2$ of $\mathcal{G}_{1,3}$ are $e_{14}, e_{24}, e_{34}, ie_{14}, ie_{24}, ie_{34}$, where $i = e_{1234}$ is the *pseudoscalar* of $\mathcal{G}_{1,3}$. Note that the first 3 of these bivectors have square $+1$, where as the *duals* of these basis bivectors have square -1. Indeed, the subalgebra of $\mathcal{G}_{1,3}$ generated by $E_1 = e_{41}, E_2 = e_{42}, E_3 = e_{43}$ is algebraically isomorphic to the geometric algebra $\mathcal{G}_3$ of space. This key relationship makes possible the efficient expression of electromagnetism and the theory of special relativity in one and the same formalisms.

An important class of pseudo-euclidean spaces consists of those that have *neutral signature*, $\mathcal{G}_{n,n} = \mathcal{G}_{n,n}(I\!R^{n,n})$. The simplest such algebra is $\mathcal{G}_{1,1}$ with the standard basis $\{1, e_1, e_2, e_{12}\}$, where $e_1^2 = 1 = -e_2^2$ and $e_{12}^2 = 1$. We shall shortly see that $\mathcal{G}_{1,1}$ is the basic building block for extending the applications of geometric algebra to affine and projective and other non-euclidean geometries, and for exploring the structure of geometric algebras in terms of matrices.

1.4 Spectral basis and matrices of geometric algebras

Until now we have only discussed the standard basis of a geometric algebra $\mathcal{G}$. The standard basis is very useful for presenting the basic rules of the algebra and its geometric interpretation as a graded algebra of multivectors of different grades, a k-blade characterizing the direction of a k-dimensional subspace. There is another basis for a geometric algebra, called a *spectral basis* , that is very useful for relating the structure of a geometric algebra to corresponding isomorphic matrix algebras [15]. Another term that has been applied is *spinor basis* , but I prefer the term "spectral basis" because of its deep roots in linear algebra [17].

The key to constructing a spectral basis for any geometric algebra $\mathcal{G}$ is to pick out any two elements $u, v \in \mathcal{G}$ such that $u^2 = 1$, v^{-1} exists, and $uv = -vu \neq 0$. We then define the *idempotents* $u_+ = \frac{1}{2}(1+u)$ and $u_- = \frac{1}{2}(1-u)$ in $\mathcal{G}$, and note that

$$u_+^2 = u_+, u_-^2 = u_-, u_+u_- = u_-u_+ = 0, \text{ and } u_+ + u_- = 1.$$

We say that $u_\pm$ are *mutually annihiliating idempotents which partition 1.* Also $vu_+ = u_-v$, from which it follows that $v^{-1}u_+ = u_-v^{-1}$.

Using these simple algebraic properties, we can now *factor out* a 2×2 matrix algebra from $\mathcal{G}$. Adopting matrix notation, first note that

$$\begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix} = \begin{pmatrix} u_+ & vu_+ \end{pmatrix} \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix} = u_+ + u_- = 1.$$

For any element $g \in \mathcal{G}$, we have

$$\begin{aligned} g &= \begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix} g \begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} g & gv \\ v^{-1}g & v^{-1}gv \end{pmatrix} u_+ \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix}. \end{aligned} \tag{11}$$

The expression $[g] := u_+ \begin{pmatrix} g & gv \\ v^{-1}g & v^{-1}gv \end{pmatrix} u_+$ is called the *matrix decomposition* of g with respect to the elements $\{u, v\} \subset \mathcal{G}$. To see that the mapping $g \mapsto [g]$ gives a *matrix isomorphism*, in the sense that $[g+h] = [g] + [h]$ and $[gh] = [g][h]$ for all $g, h \in \mathcal{G}$, it is obvious that we only need to check the multiplicative property. We find that

$$[g][h] = u_+ \begin{pmatrix} g & gv \\ v^{-1}g & v^{-1}gv \end{pmatrix} u_+ \begin{pmatrix} h & hv \\ v^{-1}h & v^{-1}hv \end{pmatrix} u_+$$

$$= u_+ \begin{pmatrix} gu_+h + gvu_+v^{-1}h & gu_+hv + gvu_+v^{-1}hv \\ v^{-1}gu_+h + v^{-1}gvu_+v^{-1}h & v^{-1}gu_+hv + v^{-1}gvu_+v^{-1}hv \end{pmatrix} u_+$$

$$= u_+ \begin{pmatrix} gh & ghv \\ v^{-1}gh & v^{-1}ghv \end{pmatrix} u_+ = [gh].$$

To fully understand the nature of this matrix isomorphism, we need to know about the nature of the entries of $[g]$ in (11). We will analyse the special case where $v^2 \in I\!R$, although the relationship is valid for more general v. In this case, the entries of $[g]$ can be decomposed in terms of *conjugations* with respect to the elements u and v. Let $a \in \mathcal{G}$ for which a^{-1} exists. The *a-conjugate* $\overline{g}^a$ of the element $g \in \mathcal{G}$ is defined by $\overline{g}^a = aga^{-1}$.

We shall use u- and v-conjugates to decompose any element g into the form

$$g = G_1 + uG_2 + v(G_3 + uG_4) \tag{12}$$

where $G_i \in C_{\mathcal{G}}(\{u,v\})$, the subalgebra of all elements of $\mathcal{G}$ which commute with the subalgebra generated by $\{u,v\}$. It follows that $\mathcal{G} = C_{\mathcal{G}}(\{u,v\}) \otimes \{u,v\}$ or $\mathcal{G} \equiv \mathcal{M}_2(C_{\mathcal{G}}(\{u,v\}))$. This means that $\mathcal{G}$ is isomorphic to a 2×2 matrix algebra over $C_{\mathcal{G}}(\{u,v\})$.

We first decompose g into the form

$$g = \frac{1}{2}(g + \overline{g}^u) + v[\frac{v^{-1}}{2}(g - \overline{g}^u)] = g_1 + vg_2$$

where $g_1, g_2 \in C_{\mathcal{G}}(u)$, the geometric subalgebra of $\mathcal{G}$ of all elements which commute with the element u. By further decomposing g_1 and g_2 with respect to the v-conjugate, we obtain the decomposition

$$g = G_1 + uG_2 + v(G_3 + uG_4)$$

where each $G_i \in C_{\mathcal{G}}(\{u,v\})$. Specifically, we have

$$G_1 = \frac{1}{2}(g_1 + \overline{g}_1^v) = \frac{1}{4}(g + ugu + vgv^{-1} + vuguv^{-1})$$

$$G_2 = \frac{u}{2}(g_1 - \overline{g}_1^v) = \frac{u}{4}(g + ugu - vgv^{-1} - vuguv^{-1})$$

$$G_3 = \frac{1}{2}(g_2 + \overline{g}_2^v) = \frac{v^{-1}}{4}(g - ugu + vgv^{-1} - vuguv^{-1})$$

$$G_4 = \frac{u}{2}(g_2 - \overline{g}_2^v) = \frac{uv^{-1}}{4}(g - ugu - vgv^{-1} + vuguv^{-1}).$$

Using the decomposition (12) of g, we find the 2×2 matrix decomposition (11) of g over the module $C_{\mathcal{G}}(\{u,v\})$,

$$[g] := u_+ \begin{pmatrix} g & gv \\ v^{-1}g & v^{-1}gv \end{pmatrix} u_+ = u_+ \begin{pmatrix} G_1 + G_2 & v^2(G_3 - G_4) \\ G_3 + G_4 & G_1 - G_2 \end{pmatrix} \tag{13}$$

where $G_i \in C_{\mathcal{G}}(\{u,v\})$ for $1 \le i \le 4$.

For example, for $g \in \mathcal{G}_3$ and $u = e_1$, $v = e_{12} = -v^{-1}$, we write $g = (z_1 + uz_2) + v(z_3 + uz_4)$, where $z_j = x_j + iy_j$ for $i = e_{123}$ and $1 \le j \le 4$. Noting that $u_\pm u = uu_\pm = \pm u_\pm$ and $u_\pm v = vu_\mp$, and substituting this complex form of g into the above equation gives

$$g = \begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} g & gv \\ v^{-1}g & v^{-1}gv \end{pmatrix} u_+ \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & v \end{pmatrix} u_+ \begin{pmatrix} z_1 + z_2 & z_4 - z_3 \\ z_4 + z_3 & z_1 - z_2 \end{pmatrix} \begin{pmatrix} 1 \\ v^{-1} \end{pmatrix}.$$

We say that

$$[g] = u_+ \begin{pmatrix} z_1 + z_2 & z_4 - z_3 \\ z_4 + z_3 & z_1 - z_2 \end{pmatrix}$$

is the matrix decomposition of $g \in \mathcal{G}_3$ over the complex numbers, $\mathbb{C} = \{x + iy\}$ where $i = e_{123}$. It follows that

$$\mathcal{G}_3 \equiv \mathcal{M}_2(\mathbb{C}).$$

There are many decompositions of Clifford geometric algebras into isomorphic matrix algebras. As shown in the example above, a matrix decomposition of geometric algebra is equivalent to selecting a *spectral basis*, in this case

$$\begin{pmatrix} 1 \\ v \end{pmatrix} u_+ \begin{pmatrix} 1 & v^{-1} \end{pmatrix} = \begin{pmatrix} u_+ & v^{-1}u_- \\ vu_+ & u_- \end{pmatrix},$$

as opposed to the standard basis for the algebra. The relative position of the elements in the spectral basis, written as a matrix above, gives the isomorphism between the geometric algebra and the matrix algebra.

There is a matrix decomposition of the geometric algebra $\mathcal{G}_{p+1,q+1}$ that is very useful. For this decomposition we let $u = e_{p+1}e_{p+q+2}$, so that the bivector u has the property that $u^2 = 1$, and let $v = e_{p+1}$. We then have the idempotents $u_\pm = \frac{1}{2}(1 \pm u)$, satisfying $vu_\pm = u_\mp v$, and giving the decomposition

$$\mathcal{G}_{p+1,q+1} = \mathcal{G}_{1,1} \otimes \mathcal{G}_{p,q} \equiv \mathcal{M}_2(\mathcal{G}_{p,q}) \tag{14}$$

for $\mathcal{G}_{1,1} = gen\{e_{p+1}, e_{p+q+2}\}$ and $\mathcal{G}_{p,q} = gen\{e_1, \ldots, e_p, e_{p+2}, \ldots, e_{p+q+1}\}$.

2. Projective Geometries

Leonardo da Vinci (1452–1519) was one of the first to consider the problems of projective geometry. However, projective geometry was not formally developed until the work "Traité des propriés projectives des figure" of the French mathematician Poncelet (1788-1867), published in 1822. The extrordinary generality and simplicity of projective geometry led the English mathematician Cayley to exclaim: "Projective Geometry is all of geometry" [18].

The projective plane is almost identical to the Euclidean plane, except for the addition of ideal points and an ideal line at infinity. It seems natural, therefore, that in the study of analytic projective geometry the coordinate systems of Euclidean plane geometry should be almost sufficient. It is also required that these ideal objects at infinity should be indistinquishable from their corresponding ordinary objects, in this case ordinary points and ordinary lines. The solution to this problem is the introduction of "homogeneous coordinates", [6, p. 71]. The introduction of the tools of homogeneous coordinates is accomplished in a very efficient way using geometric algebra [9]. While the definition of geometric algebra does indeed involve a metric, that fact in no way prevents it from being used as a powerful tool to solve the metric-free results of projective geometry. Indeed, once the objects of projective geometry are identified with the corresponding objects of linear algebra, the whole of the machinery of geometric algebra applied to linear algebra can be carried over to projective geometry.

Let $I\!R^{n+1}$ be an $(n+1)$-dimensional euclidean space and let $\mathcal{G}_{n+1}$ be the corresponding geometric algebra. The *directions* or *rays* of non-zero vectors in $I\!R^{n+1}$ are identified with the points of the n-dimensional projective plane Π^n. More precisely, we write

$$\Pi^n \equiv I\!R^{n+1}/I\!R^*$$

where $I\!R^* = I\!R - \{0\}$. We thus identify *points*, *lines*, *planes*, and higher dimensional *k-planes* in Π^n with 1, 2, 3, and $(k+1)$-dimensional subspaces $\mathcal{S}^{k+1}$ of $I\!R^{n+1}$, where $k \le n$. To effectively apply the tools of geometric algebra, we need to introduce the basic operations of *meet* and *join*.

2.1 The Meet and Join Operations

The *meet* and *join* operations of projective geometry are most easily defined in terms of the *intersection* and *union* of the linear subspaces which name the objects in Π^n. Each r-dimensional subspace $\mathcal{A}^r$ is described by a non-zero r-blade $A_r \in \mathcal{G}(I\!R^{n+1})$. We say that an r-blade $A_r \ne 0$ *represents*, or is a *representant* of an r-subspace $\mathcal{A}^r$ of $I\!R^{n+1}$ if and only if

$$\mathcal{A}^r = \{x \in I\!R^{n+1} |\ \ x \wedge A_r = 0\}. \tag{15}$$

The equivalence class of all nonzero r-blades $A_r \in \mathcal{G}(I\!R^{n+1})$ which define the subspace $\mathcal{A}^r$ is denoted by

$$\{A_r\}_{ray} := \{tA_r \mid t \in I\!R,\ t \neq 0\}. \tag{16}$$

Evidently, every r-blade in $\{A_r\}_{ray}$ is a representant of the subspace $\mathcal{A}^r$. With these definitions, the problem of finding the meet and join is reduced to the problem of finding the corresponding *meet* and *join* of the $(r+1)$- and $(s+1)$-blades in the geometric algebra $\mathcal{G}(I\!R^{n+1})$ which represent these subspaces.

Let A_r, B_s and C_t be non-zero blades representing the three subspaces $\mathcal{A}^r$, $\mathcal{B}^s$ and $\mathcal{C}^t$, respectively. Following [15], we say that

Definition 3 *The t-blade $C_t = A_r \cap B_s$ is the meet of A_r and B_s if there exists a complementary $(r-t)$-blade A_c and a complementary $(s-t)$-blade B_c with the property that $A_r = A_c \wedge C_t$, $B_s = C_t \wedge B_c$, and $A_c \wedge B_c \neq 0$.*

It is important to note that the t-blade $C_t \in \{C_t\}_{ray}$ is not unique and is defined only up to a non-zero scalar factor, which we choose at our own convenience. The existence of the t-blade C_t (and the corresponding complementary blades A_c and B_c) is an expression of the basic relationships that exists between linear subspaces.

Definition 4 *The $(r+s-t)$-blade $D = A_r \cup B_s$, called the join of A_r and B_s is defined by $D = A_r \cup B_s = A_r \wedge B_c$.*

Alternatively, since the join $A_r \cup B_s$ is defined only up to a non-zero scalar factor, we could equally well define D by $D = A_c \wedge B_s$. We use the symbols $\cap$ *intersection* and $\cup$ *union* from set theory to mark this unusual state of affairs. The problem of "meet" and "join" has thus been solved by finding the direct sum and intersection of linear subspaces and their $(r+s-t)$-blade and t-blade representants.

Note that it is only in the special case when $A_r \cap B_s = 0$ that the join can be considered to *reduce* to the outer product. That is

$$A_r \cap B_s = 0 \quad \Leftrightarrow \quad A_r \cup B_s = A_r \wedge B_s.$$

In any case, once the join $J := A_r \cup B_s$ has been found, it can be used to find the meet

$$A_r \cap B_s = A_r \cdot [B_s \cdot J] = [JJA_r] \cdot [B_s \cdot J] = [(A_r \cdot J) \wedge (B_s \cdot J)] \cdot J \tag{17}$$

In the case that $J = I^{-1}$, we can express this last relationship in terms of the operation of duality defined in (10), $A_r \cap B_s = (A_r^* \wedge B_s^*)^* =$

$(A_r^* \cup B_s^*)^*$ which is DeMorgan's formula. It must always be remembered that the "equalities" in these formulas only mean "up to a non-zero real number". While the positive definite metric of $\mathbb{R}^{n+1}$ is irrelevant to the definition of the meet and join of subspaces, the formula (17) holds only in $\mathbb{R}^{n+1}$.

A slightly modified version of this formula will hold in any non-degenerate pseudo-euclidean space $\mathbb{R}^{p,q}$ and its corresponding geometric algebra $\mathcal{G}_{p,q} := \mathcal{G}(\mathbb{R}^{p,q})$, where $p+q = n+1$. In this case, after we have found the join $J = A_r \cup B_s$, we first find any blade $\overline{J}$ of the same step which satisfies the property that $\overline{J} \cdot J = 1$. The blade $\overline{J}$ is called a *reciprocal blade* of the blade J in the geometric algebra $\mathcal{G}_{p,q}$. The meet $A_r \cap B_s$ may then be defined by

$$A_r \cap B_s = A_r \cdot [B_s \cdot \overline{J}] = [(A_r \cdot \overline{J}) \cdot J] \cdot [B_s \cdot \overline{J}] = \{[(A_r \cdot \overline{J}) \wedge (B_s \cdot \overline{J})]\} \cdot J \quad (18)$$

The meet and join operations formulated in geometric algebra can be used to efficiently prove the many famous theorems of projective geometry [9]. See also *Geometric-Affine-Projective Computing* at the website [12].

2.2 Incidence, Projectivity and Colineation

Let $J \in \mathcal{G}_{p,q}^{k+1}$ be a $(k+1)$-blade representing a projective k-dimensional subplane in Π^n where $k \leq n$. A point (ray) $x \in \Pi^n$ is said to be *incident* to J if and only if $x \wedge J = 0$. Since J is a $(k+1)$-blade, we can find vectors $a_1, \ldots, a_{k+1} \in \mathbb{R}^{p,q}$ such that $J = a_1 \wedge \cdots \wedge a_{k+1}$. Projectively speaking, this means we can find $k+1$ non-co$(k-1)$planar points a_i in the k-projective plane J.

Now let $\overline{J}$ be a reciprocal blade to J with the property that $J \cdot \overline{J} = 1$. With the help of $\overline{J}$, we can define a *determinant function* or *bracket* $[\cdots]_{\overline{J}}$ on the projective k-plane J. Let $b_1, \ldots, b_{k+1}$ be $(k+1)$ points incident to J,

$$[b_1, \cdots, b_{k+1}]_{\overline{J}} := (b_1 \wedge \cdots \wedge b_{k+1}) \cdot \overline{J}. \quad (19)$$

The bracket $[b_1, \cdots, b_{k+1}]_{\overline{J}} \neq 0$ iff the points b_i are not co-$(k-1)$planar.

We now give the definitions necessary to complete the translation of real projective geometry into the language of multilinear algebra as formulated in geometric algebra.

DEFINITION 5 *A central perspectivity is a transformation of the points of a line onto the points of a line for which each pair of corresponding points is collinear with a fixed point called the center of perspectivity. See Figure 4.*

The key idea in the analytic expression of projective geometry in geometric algebra is that to each projectivity in Π^n there corresponds a non-singular linear transformation[2] $T : \mathbb{R}^{p,q} \longrightarrow \mathbb{R}^{p,q}$. It is clear that each projectivity of *points* on Π^n induces a corresponding *projective collineation* of lines, of planes, and higher dimensional projective k-planes. The corresponding extension of the linear transformation T from $\mathbb{R}^{p,q}$ to the whole geometric algebra $\mathcal{G}_{p,q}$ which accomplishes this is called the *outermorphism* $\mathbf{T} : \mathcal{G}_{p,q} \longrightarrow \mathcal{G}_{p,q}$, which is defined in terms of T by the properties:

$$\mathbf{T}(1) := 1, \quad \mathbf{T}(x) = T(x), \quad \mathbf{T}(x_1 \wedge \cdots \wedge x_k) := T(x_1) \wedge \cdots \wedge T(x_k) \quad (20)$$

for each $2 \leq k \leq p+q$, and then extended linearly to all elements of $\mathcal{G}_{p,q}$. Outermorphisms in geometric algebra, first studied in [16], provide the backbone for the application of geometric algebra to linear algebra. Since in everything that follows we will be using the outermorphism $\mathbf{T}$ defined by T, we will drop the boldface notation and simply use the same symbol T for both the linear transformation and its extension to an outermorphism $\mathbf{T}$.

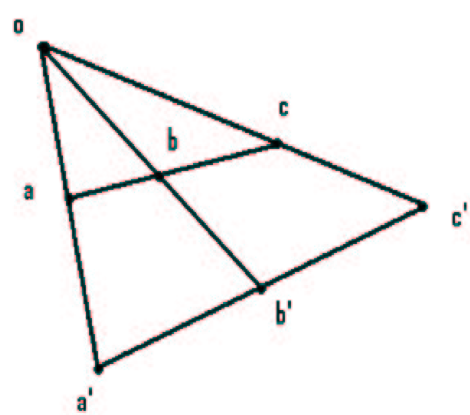

Figure 4. A central perspectivity from the point $\mathbf{o}$.

[2]Unique up to a non-zero scalar factor.

DEFINITION 6 *A projective transformation or projectivity is a transformation of points of a line onto the points of a line which may be expressed as a finite product of central perspectivities.*

We can now easily prove

THEOREM 1 *There is a one-one correspondence between non-singular outermorphisms $T : \mathcal{G}_{p,q} \longrightarrow \mathcal{G}_{p,q}$, and projective collineations on Π^n taking $n+1$ non-co$(n-1)$planar points in Π^n into $n+1$ non-co(n-1)planar points in Π^n.*

Proof: Let $a_1, \ldots, a_{n+1} \in \Pi^n$ be $n+1$ non-co(n-1)planar points. Since they are non-co(n-1)planar, it follows that $a_1 \wedge \cdots \wedge a_{n+1} \neq 0$. Suppose that $b_i = T(a_i)$ is a projective transformation between these points for $1 \leq i \leq n+1$. The corresponding non-singular outermorphism is defined by considering T to be a linear transformation on the basis vectors $a_1, \ldots, a_{n+1}$ of $I\!R^{p,q}$. Conversely, if a non-singular outermorphism is specified on $I\!R^{p,q}$ it clearly defines a unique projective collineation on Π^n, which we denote by the same symbol T.

All of the theorems on harmonic points and cross ratios of points on a projective line follow easily from the above definitions and properties [9], but we will not prove them here. For what follows, we will need two more definitions:

DEFINITION 7 *A nonidentity projectivity of a line onto itself is elliptic, parabolic, or hyperbolic as it has no, one, or two fixed points, respectively. More generally, we will say that a nonidentity projectivity of Π^n is elliptic, parabolic, or hyperbolic, if whenever it fixes a line in Π^n, then the restriction to each such line is elliptic, parabolic, or hyperbolic, respectively.*

Let $a, b \in \Pi^n$ be distinct points so that $a \wedge b \neq 0$. If T is a projectivity of Π^n and $T(a \wedge b) = \lambda a \wedge b$ for $\lambda \in I\!R^*$, then the characteristic equation of T restricted to the subspace $a \wedge b$,

$$[(\lambda - T)(a \wedge b)] \cdot \overline{a \wedge b} = 0 \qquad (21)$$

will have 0, 1 or 2 real roots, according to whether T has 0, 1 or 2 real eigenvectors, [15], [8, p.73], which correspond directly to fixed points.

DEFINITION 8 *A nonidentity projective transformation T of a line onto itself is an involution if $T^2 =$ identity.*

2.3 Conics and Polars

Let $a_1, a_2, \ldots, a_{n+1} \in I\!R^{p,q}$ represent $n+1 = p+q$ linearly independent vectors in $I\!R^{p,q}$. This means that $I = a_1 \wedge a_2 \wedge \cdots \wedge a_{n+1} \neq 0$. As an element in the projective space Π^n, I represents the projective n-plane determined by the $n+1$ non-co$(n-1)$planar points $a_1, \ldots, a_{n+1}$. Representing the points of Π^n by homogeneous vectors $x \in \mathbb{R}^{p,q}$, makes it easy to study the *quadric* hypersurface (conic) Q in Π^n defined by

$$Q := \{x| \ \ x \in \mathbb{R}^{p+q}, x \neq 0, \text{ and } x^2 = 0\}. \tag{22}$$

DEFINITION 9 *The polar of the k-blade $A \in \mathcal{G}_{p,q}^k$ is the $(n+1-k)$-blade $Pol_Q(A)$ defined by*

$$Pol_Q(A) := AI^{-1} = A^* \tag{23}$$

where A^* is the dual of A in the geometric algebra $\mathcal{G}_{p,q}$.

The above definition shows that *polarization* in the quadric hypersurface Q and *dualization* in the geometric algebra $\mathcal{G}_{p,q}$ are identical operations.

If $x \in Q$, it follows that

$$x \wedge Pol_Q(x) = x \wedge (xI^{-1}) = x^2 I^{-1} = 0. \tag{24}$$

This tells us that $Pol_Q(x)$ is the hyperplane which is tangent to Q at the point x, [12]. We will meet something very similar when we discuss the *horosphere* in section 5.

3. Affine and other geometries

In this section, we explore in what sense "projective geometry is all of geometry" as exclaimed by Cayley. In order to keep the discussion as simple as possible, we will discuss the relationship of the 2 dimensional projective plane to other 2 dimensional planar geometries, [6].

We begin with affine geometry of the plane. Let Π^2 be the real projective plane, and $\mathcal{T}(\Pi^2)$ the group of all projective transformations on Π^2. Let a line $\mathcal{L} \in \Pi^2$ (a degenerate conic) be picked out as the *absolute line*, or the *line at infinity.*

DEFINITION 10 *The affine plane $\mathcal{A}^2$ consists of all points of the projective plane Π^2 with the points on the absolute line deleted. The projective transformations leaving $\mathcal{L}$ fixed, restricted to the real affine plane, are real affine transformations. The study of $\mathcal{A}^2$ and the subgroup of real affine transformations $\mathcal{T}\{\mathcal{A}^2\}$ is real plane affine geometry.*

If we now fix an elliptic involution, called the *absolute involution*, on the line $\mathcal{L}$, then a real affine transformation which leaves the absolute involution invariant is called a *similarity* transformation.

DEFINITION 11 *The study of the group of similarity transformations on the affine plane is similarity geometry.*

An affine transformation which leaves the *area* of all triangles invariant is called an equiareal transformation.

Definition 12 *The study of the affine plane under equiareal transformations is equiareal geometry.*

A *euclidean transformation* on the affine plane is an affine transformation which is both a similarity and an equiareal transformation. Finally, we have

Definition 13 *The study of the affine plane under euclidean transformations is euclidean geometry.*

In our representation of Π^n in a geometric algebra $\mathcal{G}_{p,q}$ where $n+1 = p+q$, a projectivity is represented by a non-singular linear transformation. Thus, the group of projectivities becomes just the *general linear group* of all non-singular transformations on $\mathbb{R}^{p,q}$ extended to outermorphisms on $\mathcal{G}_{p,q}$. Generally, we may choose to work in a euclidean space $\mathbb{R}^{n+1}$, rather than in the pseudo-euclidean space $\mathbb{R}^{p,q}$, [9], [12]. We have chosen here to work in the more general geometric algebra $\mathcal{G}_{p,q}$, because of the more direct connection to the study of a particular nondegenerate quadric hypersurface or conic in $\mathbb{R}^{p,q}$.

We still have not mentioned two important classes of non-euclidean plane geometries, *hyperbolic plane geometry* and *elliptic plane geometry* , and their relation to projective geometry. Unlike euclidean geometry, where we picked out a degenerate conic called the absolute line or line at infinity, the study of these other types of plane geometries involves the picking out of a nondegenerate conic called the *absolute conic* .

For plane hyperbolic geometry, we pick out a real non-degenerate conic in Π^2, called the *absolute conic*, and define the points interior to the absolute to be *ordinary* , those points on the absolute are *ideal* , and those points exterior to the absolute are *ultraideal* [6, p.230].

Definition 14 *A real projective plane from which the absolute conic and its exterior have been deleted is a hyperbolic plane. The projective collineations leaving the absolute fixed and carrying interior points onto interior points, restricted to the hyperbolic plane, are hyperbolic isometries. The study of the hyperboic plane and hyperbolic isometries is hyperbolic geometry.*

Hyperbolic geometry has been studied extensively in geometric algebra by Hongbo Li in [3, p.61-85], and applied to automatic theorem proving in [3, p.110-119], [5, p.69-90].

For plane elliptic geometry, we pick out an *imaginary* nondegenerate conic in Π^2 as the *absolute conic*. Since there are no real points on this conic, the points of elliptic geometry are the same as the points in the real projective plane Π^2. A projective collineation which leaves the absolute conic fixed (whose points are in the complex projective plane) is called an *elliptic isometry* .

DEFINITION 15 *The real projective plane* Π^2 *is the elliptic plane. The study of the elliptic plane and elliptic isometries is elliptic geometry.*

In the next section, we return to the study of affine geometries of higher dimensional pseudo-euclidean spaces. However, we shall not study the formal properties of these spaces. Rather, our objective is to efficiently define the *horosphere* of a pseudo-euclidean space, and study some of its properties.

4. Affine Geometry of pseudo-euclidean space

We have seen that a projective space can be considered to be an affine space with idealized points at infinity [18]. Since all the formulas for meet and join remain valid in the pseudo-euclidean space $I\!R^{p,q}$, using (18), we define the $n=(p+q)$-dimensional affine plane $\mathcal{A}_e(I\!R^{p,q})$ of the null vector $e=\frac{1}{2}(\sigma+\eta)$ in the larger pseudo-euclidean space $I\!R^{p+1,q+1}=I\!R^{p,q}\oplus I\!R^{1,1}$, where $I\!R^{1,1}=span\{\sigma,\eta\}$ for $\sigma^2=1=-\eta^2$. Whereas, effectively, we are only extending the euclidean space $I\!R^{p,q}$ by the null vector e, it is advantageous to work in the geometric algebra $\mathcal{G}_{p+1,q+1}$ of the *non-degenerate* pseudo-euclidean space $I\!R^{p+1,q+1}$. We give here the important properties of the reciprocal null vectors $e=\frac{1}{2}(\sigma+\eta)$ and $\overline{e}=\sigma-\eta$ that will be needed later, and their relationship to the *hyperbolic unit bivector* $u:=\sigma\eta$.

$$e^2=\overline{e}^2=0,\ e\cdot\overline{e}=1,\ u=\overline{e}\wedge e=\sigma\wedge\eta,\ u^2=1. \tag{25}$$

The affine plane $\mathcal{A}_e^{p,q}:=\mathcal{A}_e(I\!R^{p,q})$ is defined by

$$\mathcal{A}_e(I\!R^{p,q})=\{x_h=x+e|\ \ x\in I\!R^{p,q}\ \}\subset I\!R^{p+1,q+1}, \tag{26}$$

for the null vector $e\in I\!R^{1,1}$. The affine plane $\mathcal{A}_e(I\!R^{p,q})$ has the nice property that $x_h^2=x^2$ for all $x_h\in\mathcal{A}_e(I\!R^{p,q})$, thus preserving the metric structure of $I\!R^{p,q}$. We can restate definition (26) of $\mathcal{A}_e(I\!R^{p,q})$ in the form

$$\mathcal{A}_e(I\!R^{p,q})=\{y|\ \ y\in I\!R^{p+1,q+1}\ ,\quad y\cdot\overline{e}=1\quad\text{and}\quad y\cdot e=0\ \}\subset I\!R^{p+1,q+1}.$$

This form of the definition is interesting because it brings us closer to the definition of the $n=(p+q)$-dimensional *projective plane* .

The projective n-plane Π^n can be defined to be the set of all points of the affine plane $\mathcal{A}_e(I\!R^{p,q})$, taken together with idealized points at infinity. Each point $x_h \in \mathcal{A}_e(I\!R^{p,q})$ is called a *homogeneous representant* of the corresponding point in Π^n because it satisfies the property that $x_h \cdot \overline{e} = 1$. To bring these different viewpoints closer together, points in the affine plane $\mathcal{A}_e(I\!R^{p,q})$ will also be represented by *rays* in the space

$$\mathcal{A}_e^{rays}(I\!R^{p,q}) = \{\{y\}_{ray} |\ y \in I\!R^{p+1,q+1}, y \cdot e = 0, \quad y \cdot \overline{e} \neq 0 \ \} \subset I\!R^{p+1,q+1}. \tag{27}$$

The set of rays $\mathcal{A}_e^{rays}(I\!R^{p,q})$ gives another definition of the affine n-plane, because each ray $\{y\}_{ray} \in \mathcal{A}_e^{rays}(I\!R^{p,q})$ determines the unique homogeneous point

$$y_h = \frac{y}{y \cdot \overline{e}} \in \mathcal{A}_e(I\!R^{p,q}).$$

Conversely, each point $y \in \mathcal{A}_e(I\!R^{p,q})$ determines a unique ray $\{y\}_{ray}$ in $\mathcal{A}_e^{rays}(I\!R^{p,q})$. Thus, the affine plane of homogeneous points $\mathcal{A}_e(I\!R^{p,q})$ is equivalent to the affine plane of rays $\mathcal{A}_e^{rays}(I\!R^{p,q})$.

Suppose that we are given k-points $a_1^h, a_2^h, \ldots, a_k^h \in \mathcal{A}_e(I\!R^{p,q})$ where each $a_i^h = a_i + e$ for $a_i \in I\!R^{p,q}$. Taking the outer product or *join* of these points gives the projective $(k-1)$-plane $A^h \in \Pi^n$. Expanding the outer product gives

$$\begin{aligned} A^h &= a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h = a_1^h \wedge (a_2^h - a_1^h) \wedge a_3^h \wedge \ldots \wedge a_k^h \\ &= a_1^h \wedge (a_2^h - a_1^h) \wedge (a_3^h - a_2^h) \wedge a_4^h \wedge \ldots \wedge a_k^h = \ldots \\ &= a_1^h \wedge (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}), \end{aligned}$$

or

$$\begin{aligned} A^h &= a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h = a_1 \wedge a_2 \wedge \ldots \wedge a_k + \\ & e \wedge (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}). \end{aligned} \tag{28}$$

Whereas (28) represents a $(k-1)$-plane in Π^n, it also belongs to the affine (p,q)-plane $\mathcal{A}_e^{p,q}$, and thus contains important metrical information. Dotting this equation with $\overline{e}$, we find that

$$\overline{e} \cdot A^h = \overline{e} \cdot (a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h) = (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}).$$

This result motivates the following

Definition 16 *The directed content of the $(k-1)$-simplex*

$$A^h = a_1^h \wedge a_2^h \wedge \cdots \wedge a_k^h$$

in the affine (p,q)-plane is given by

$$\begin{aligned} \frac{\overline{e} \cdot A^h}{(k-1)!} &= \frac{\overline{e} \cdot (a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h)}{(k-1)!} \\ &= \frac{(a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1})}{(k-1)!}. \end{aligned}$$

4.1 Example

Many incidence relations can be expressed in the affine plane $\mathcal{A}_e(\mathbb{R}^{p,q})$ which are also valid in the projective plane Π^n, [3, pp.263]. We will only give here the simplest example.

Given are 4 coplanar points $a_h, b_h, c_h, d_h \in \mathcal{A}_e(\mathbb{R}^2)$. The join and meet of the lines $a_h \wedge b_h$ and $c_h \wedge d_h$ are given, respectively, by $(a_h \wedge b_h) \cup (c_h \wedge d_h) = a_h \wedge b_h \wedge c_h$, and using (18),

$$(a_h \wedge b_h) \cap (c_h \wedge d_h) = [\overline{I} \cdot (a_h \wedge b_h)] \cdot (c_h \wedge d_h)$$

where e_1, e_2 are the orthonormal basis vectors of $\mathbb{R}^2$, and $\overline{I} = e_2 \wedge e_1 \wedge \overline{e}$. Carrying out the calculations for the meet and join in terms of the bracket determinant (19), we find that

$$(a_h \wedge b_h) \cup (c_h \wedge d_h) = [a_h, b_h, c_h]_{\overline{I}} I = \det\{a, b\} I \tag{29}$$

where $I = e_1 \wedge e_2 \wedge e$ and $\det\{a, b\} := (a \wedge b) \cdot (e_{21})$, and

$$(a_h \wedge b_h) \cap (c_h \wedge d_h) = \det\{c - d, b - c\} a_h + \det\{c - d, c - a\} b_h. \tag{30}$$

Note that the meet (30) is not, in general, a homogeneous point. Normalizing (30), we find the homogeneous point $p_h \in \mathcal{A}_e(\mathbb{R}^2)$

$$p_h = \frac{\det\{c - d, b - c\} a_h + \det\{c - d, c - a\} b_h}{\det\{c - d, b - a\}}$$

which is the intersection of the lines $a_h \wedge b_h$ and $c_h \wedge d_h$. The meet can also be solved for directly in the affine plane by noting that

$$p_h = \alpha_p a_h + (1 - \alpha_p) b_h = \beta_p c_h + (1 - \beta_p) d_h$$

and solving to get $\alpha_p = [b_h, c_h, d_h]_{\overline{I}} / [b_h - a_h, c_h, d_h]_{\overline{I}}$. Other simple examples can be found in [15].

5. Conformal Geometry and the Horosphere

The conformal geometry of a pseudo-Euclidean space can be linearized by considering the *horosphere* in a pseudo-Euclidean space of two dimensions higher. We begin by defining the *horosphere* $\mathcal{H}_e^{p,q}$ in $\mathbb{R}^{p+1,q+1}$ by *moving up* from the affine plane $\mathcal{A}_e^{p,q} := \mathcal{A}_e(\mathbb{R}^{p,q})$.

5.1 The horosphere

Let $\mathcal{G}_{p+1,q+1} = gen(\mathbb{R}^{p+1,q+1})$ be the geometric algebra of $\mathbb{R}^{p+1,q+1}$, and recall the definition (26) of the affine plane $\mathcal{A}_e^{p,q} := \mathcal{A}_e(\mathbb{R}^{p,q}) \subset$

$I\!R^{p+1,q+1}$. Any point $y \in I\!R^{p+1,q+1}$ can be written in the form $y = x + \alpha e + \beta\overline{e}$, where $x \in I\!R^{p,q}$ and $\alpha, \beta \in I\!R$.

The *horosphere* $\mathcal{H}_e^{p,q}$ is most directly defined by

$$\mathcal{H}_e^{p,q} := \{x_c = x_h + \beta\overline{e} \mid x_h \in \mathcal{A}_e^{p,q} \text{ and } x_c^2 = 0.\} \tag{31}$$

With the help of (25), the condition that

$$x_c^2 = (x_h + \beta\overline{e})^2 = x^2 + 2\beta = 0$$

gives us immediately that $\beta := -\frac{x^2}{2}$. Thus each point $x_c \in \mathcal{H}_e^{p,q}$ has the form

$$x_c = x_h - \frac{x_h^2}{2}\overline{e} = x + e - \frac{x^2}{2}\overline{e} = \frac{1}{2}x_h\overline{e}x_h. \tag{32}$$

The last equality on the right follows from

$$\frac{1}{2}x_h\overline{e}x_h = \frac{1}{2}[(x_h\cdot\overline{e})x_h + (x_h\wedge\overline{e})x_h] = x_h - \frac{1}{2}x_h^2\overline{e}.$$

From (32), we easily calculate

$$x_c \cdot y_c = (x + e - \frac{x^2}{2}\overline{e}) \cdot (y + e - \frac{y^2}{2}\overline{e}) =$$

$$x \cdot y - \frac{y^2}{2} - \frac{x^2}{2} = -\frac{1}{2}(x-y)^2,$$

where $(x-y)^2$ is the square of the pseudo-euclidean distance between the conformal representants x_c and y_c. We see that the pseudo-euclidean structure is preserved in the form of the inner product $x_c \cdot y_c$ on the horosphere.

Just as $x_h \in \mathcal{A}_e^{p,q}$ is called the *homogeneous representant* of $x \in I\!R^{p,q}$, the point x_c is called the *conformal representant* of *both* the points $x_h \in \mathcal{A}_e^{p,q}$ and $x \in I\!R^{p,q}$. The set of all conformal representants $\mathcal{H}^{p,q} := c(I\!R^{p,q})$ is called the *horosphere* . The horosphere $\mathcal{H}^{p,q}$ is a non-linear model of both the affine plane $\mathcal{A}_e^{p,q}$ and the pseudo-euclidean space $I\!R^{p,q}$. The horosphere $\mathcal{H}^n$ for the Euclidean space $I\!R^n$ was first introduced by F.A. Wachter, a student of Gauss, [7], and has been recently finding many diverse applications [3], [5].

The set of all null vectors $y \in I\!R^{p+1,q+1}$ make up the *null cone*

$$\mathcal{N} := \{y \in I\!R^{p+1,q+1} | \;\; y^2 = 0\}.$$

The subset of $\mathcal{N}$ containing all the representants $y \in \{x_c\}_{ray}$ for any $x \in I\!R^{p,q}$ is defined to be the set

$$\mathcal{N}_0 = \{y \in \mathcal{N} \mid y\cdot\overline{e} \neq 0\} = \cup_{x \in I\!R^{p,q}} \; \{x_c\}_{ray},$$

and is called the *restricted null cone*. The conformal representant of a null ray $\{z\}_{ray}$ is the representant $y \in \{z\}_{ray}$ which satisfies $y\cdot\overline{e} = 1$.

The horosphere $\mathcal{H}^{p,q}$ is the parabolic section of the restricted null cone,

$$\mathcal{H}^{p,q} = \{y \in \mathcal{N}_0 \mid y\cdot\overline{e} = 1\},$$

see Figure 5. Thus $\mathcal{H}^{p,q}$ has dimension $n = p + q$. The null cone $\mathcal{N}$ is determined by the condition $y^2 = 0$, which taking differentials gives

$$y\cdot dy = 0 \quad \Rightarrow \quad x_c\cdot dy = 0 \ , \tag{33}$$

where $\{y\}_{ray} = \{x_c\}_{ray}$. Since $\mathcal{N}_0$ is an $(n+1)$-dimensional surface, then (33) is a condition necessary and sufficient for a vector v to belong to the tangent space to the restricted null cone $\mathcal{T}(\mathcal{N}_0)$ at the point y

$$v \in \mathcal{T}(\mathcal{N}_0) \quad \Leftrightarrow \quad x_c\cdot v = 0 \ . \tag{34}$$

It follows that the $(n+1)$-pseudoscalar I_y of the tangent space to $\mathcal{N}_0$ at the point y can be defined by $I_y = Ix_c$ where I is the pseudoscalar of $\mathbb{R}^{p+1,q+1}$. We have

$$x_c\cdot v = 0 \quad \Leftrightarrow \quad 0 = I(x_c\cdot v) = (Ix_c)\wedge v = I_y\wedge v, \tag{35}$$

a relationship that we have already met in (24).

5.2 H-twistors

Let us define an *h-twistor* to be a rotor $S_x \in \mathrm{Spin}_{p+1,q+1}$

$$S_x := 1 + \frac{1}{2}x\overline{e} = \exp\left(\frac{1}{2}x\overline{e}\right). \tag{36}$$

An h-twistor is an equivalence class of two "twistor" components from $\mathcal{G}_{p,q}$, that have many twistor-like properties. The point x_c is generated from $0_c = e$ by

$$x_c = S_x e {S_x}^{\dagger}, \tag{37}$$

and the tangent space to the horosphere at the point x_c is generated from $dx \in \mathbb{R}^{p,q}$ by

$$dx_c = dS_x\, e\, {S_x}^{\dagger} + S_x\, e\, d{S_x}^{\dagger} = S_x(\Omega_S\cdot e){S_x}^{\dagger} = S_x dx {S_x}^{\dagger}. \tag{38}$$

It also keeps unchanged the "point at infinity" $\overline{e}$

$$\overline{e} = S_x\overline{e}{S_x}^{\dagger}.$$

H-twistors were defined and studied in [15], and more details can be found therein.

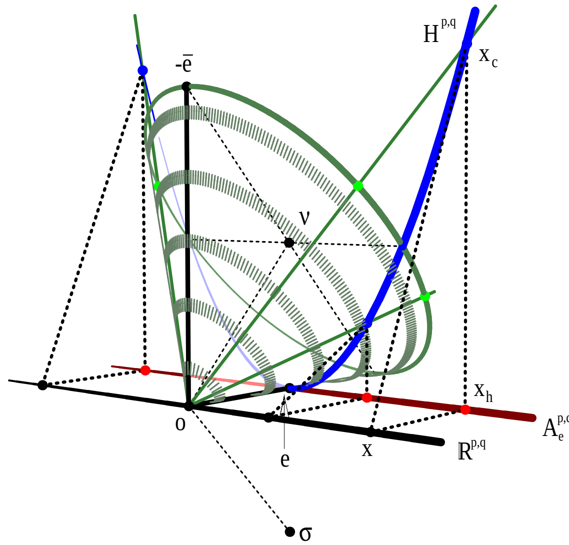

Figure 5. The restricted null cone and representations of the point x in affine space and on the horosphere.

Since the group of isometries in $\mathcal{N}_0$ is a double covering of the group of conformal transformations $Con_{p,q}$ in $I\!R^{p,q}$, and the group $Pin_{p+1,q+1}$ is a double covering of the group of orthogonal transformations $O(p{+}1, q{+}1)$, it follows that $Pin_{p+1,q+1}$ is a four-fold covering of $Con_{p,q}$, [10, p.220], [14, p.146].

5.3 Matrix representation

We have seen in (14) that the algebra $\mathcal{G}_{p+1,q+1} = \mathcal{G}_{p,q} \otimes \mathcal{G}_{1,1}$ is isomorphic to a 2×2 matrix algebra over the module $\mathcal{G}_{p,q}$. This identification makes possible a very elegant treatment of the so-called *Vahlen matrices* [10, 11, 4, 14].

Recall in section 1.4, that the idempotents $u_\pm = \frac{1}{2}(1 \pm u)$ of the algebra $\mathcal{G}_{1,1}$ satisfy the properties

$$u_+ + u_- = 1 \ , \quad u_+ - u_- = u \ , \quad u_+ u_- = 0 = u_- u_+ \ , \quad \sigma u_+ = u_- \sigma \ ,$$

where

$$u := \overline{e} \wedge e \ , \quad u_+ = \frac{1}{2}\overline{e} e \ , \quad u_- = \frac{1}{2} e \overline{e} \ ,$$

and

$$u\overline{e} = \overline{e} = -\overline{e} u \ , \quad eu = e = -ue \ , \quad \sigma u_+ = e \ , \quad 2\sigma u_- = \overline{e} \ .$$

Each multivector $G \in \mathcal{G}_{p+1,q+1}$ can be written in the form

$$G = \begin{pmatrix} 1 & \sigma \end{pmatrix} u_+ [G] \begin{pmatrix} 1 \\ \sigma \end{pmatrix} = Au_+ + Bu_+\sigma + C^* u_- \sigma + D^* u_- \tag{39}$$

where

$$[G] \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{for} \quad A, B, C, D \in \mathcal{G}_{p,q}.$$

The matrix $[G]$ denotes the matrix corresponding to the multivector G.

The operation of *reversion* of multivectors translates into the following transpose-like matrix operation:

$$\text{if} \quad [G] = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{then} \quad [G]^\dagger := [G^\dagger] = \begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix}$$

where $\overline{A} = A^{*\dagger}$ is the *Clifford conjugation*, [15].

5.4 Möbius transformations

We have seen in (37) that the point $x_c \in \mathcal{H}_{p,q}$ can be written in the form, $x_c = S_x e {S_x}^\dagger$. More generally, any conformal transformation $f(x)$ can be represented on the horosphere by

$$f(x)_c = S_{f(x)}\, e\, {S_{f(x)}}^\dagger. \tag{40}$$

Using the matrix representation (39), for a general multivector $G \in \mathcal{G}_{p+1,q+1}$ we find that

$$\begin{aligned} [GeG^\dagger] &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix} \\ &= \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} \overline{D} & \overline{B} \end{pmatrix} \end{aligned} \tag{41}$$

where

$$[e] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [G] \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad [G]^\dagger = \begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix}.$$

The relationship (41) suggests defining the *conformal h-twistor* of the multivector $G \in \mathcal{G}_{p+1,q+1}$ to be

$$[G]_c := \begin{pmatrix} B \\ D \end{pmatrix},$$

which may also be identified with the multivector $G_c := Ge = Bu_+ + D^*e$. The *conjugate* of the conformal h-twistor is then naturally defined by

$$[G]_c^\dagger := \begin{pmatrix} \overline{D} & \overline{B} \end{pmatrix} \ .$$

Conformal h-twistors give us a powerful tool for manipulating the conformal representant and conformal transformations much more efficiently. For example, since x_c in (37) is generated by the conformal h-twistor $[S_x]_c$, it follows that

$$[x_c] = [S_x]_c[S_x]_c^\dagger = \begin{pmatrix} x \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -x \end{pmatrix} = \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \ .$$

We can now write the conformal transformation (40) in its spinorial form,

$$[S_{f(x)}]_c = \begin{pmatrix} f(x) \\ 1 \end{pmatrix} .$$

Since $T_x = RS_x$ for the constant vector $R \in Pin_{p+1,q+1}$, its spinorial form is given by

$$[T_x]_c = [R][S_x]_c = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + B \\ Cx + D \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix},$$

where

$$[R] = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ , \quad \text{for constants} \quad A, B, C, D \in \mathcal{G}_{p,q}.$$

It follows that

$$[T_x] = \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} f(x) \\ 1 \end{pmatrix} H \quad \Rightarrow \quad H = N \quad \text{and} \quad f(x) = MN^{-1}. \qquad (42)$$

The beautiful linear fractional expression for the conformal transformation $f(x)$,

$$f(x) = (Ax + B)(Cx + D)^{-1} \qquad (43)$$

is a direct consequence of (42), [15].

The linear fractional expression (43) extends to any dimension and signature the well-known Möbius transformations in the complex plane. The components A, B, C, D of $[R]$ are subject to the condition that $R \in Pin_{p+1,q+1}$. Conformal h-twistors are a generalization to any dimension and any signature of the familiar 2-component spinors over the complex numbers, and the 4-component twistors. Penrose's twistor theory [13] has been discussed in the framework of Clifford algebra by a number of authors, for example see [1], [2, pp75-92].

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