On the Uncertainty Principle in Harmonic Analysis

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ABSTRACT. The Uncertainty Principle (UP) as understood in this lecture is the following informal assertion: *a non-zero "object" (a function, distribution, hyperfunction) and its Fourier image cannot be too small simultaneously*. "The smallness" is understood in a very broad sense meaning fast decay (at infinity or at a point, bilateral or unilateral), perforated (or bounded, or semibounded) support etc. The UP becomes a theorem for many "smallnesses" and has a multitude of quite concrete quantitative forms. It plays a fundamental role as one of the major themes of classical Fourier analysis (and neighboring parts of analysis), but also in applications to physics and engineering. The lecture is a review of facts and techniques related to the UP; connections with local and non-local shift invariant operators are discussed at the end of the lecture (including some topical problems of potential theory). The lecture is intended for the general audience acquainted with basic facts of Fourier analysis on the line and circle, and rudiments of complex analysis.

Introduction

This lecture is devoted to the following phenomenon known as the Uncertainty Principle (the UP):

it is impossible for a non-zero function f and its Fourier image $\hat{f}$ to be too small simultaneously. In other words, the approximate equalities $f \approx g, \hat{f} \approx \hat{g}$ cannot hold at the same time and with a high degree of accuracy unless f and g are identical. Gaining some "certainty" about f (in the form of a good approximation g) we have to pay by the uncertainty about $\hat{f}$, since the error $\hat{f} - \hat{g}$ is bound to be considerable. The term is borrowed from quantum mechanics where it is usually understood as the Heisenberg inequality for the wave function, but in the present text it is interpreted in a much less definite sense; this very vagueness makes it flexible and susceptible to a multitude of rigorous interpretations (or refutations) depending on a concrete kind of respective "smallness" of f and $\hat{f}$ mentioned in its statement. Our UP can be patently wrong (e.g. if the sizes of f and $\hat{f}$ are measured in the L^2-norm and in many other cases); this means the UP can be sometimes overcome, and "small" non-zero pairs $(f, \hat{f})$ *may* exist, this fact being also one of our themes. Nevertheless the UP plays an outstanding role in harmonic analysis and its applications to physics and engineering. But these applications won't be discussed here. We treat the UP as a phenomenon of pure mathematics, or, to be more precise, classical Fourier analysis (mainly on $\mathbb{R}$ and the unit circle $\mathbb{T}$). Our theme is very vast and can be looked at from many points of view; ours will be that of pure analysis. The concrete forms of the UP to be considered here pertain mostly to quasianalyticity, approximation theory, and, first of all, to complex analysis, an abundant source of concrete manifestations (and disprovements) of the UP. Among the omissions of this lecture are the operator theoretic approach to the UP (commutation relations) and the modern time-frequency approach. But even after we have confined our discussion to the purely analytic aspects of the phenomenon we are still left with a huge mass of facts, techniques, and approaches. Thus the choice of what is to be discussed was inescapable and difficult. It was a compromise dictated by what I know (or don't), time and size limitations, my personal predilections, but also by my desire to publicize impressive results obtained in the eighties and nineties by my (partly former) colleagues from St. Petersburg, although a good deal of the subsequent text is quite old and classical.

To describe the organization of the lecture let us first introduce some notation. Let X denote $\mathbb{R}$ or $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}; m$ will stand for Lebesgue measure on X; we always normalize m on $\mathbb{T} : m(\mathbb{T}) = 1$; sometimes we write $|A|$ in place of $m(A)$. The Fourier transform $\hat{f}$ of a function $f \in L^1 = L^1(X, m)$ is understood as $\hat{f}(\xi) = (2\pi)^{-1} \int_X f(t) \exp(-it\xi) dt$ where $\xi \in \mathbb{R}$ or $\mathbb{Z}$ (i.e. $\xi \in \hat{X}$), but different normalizations of $\hat{f}$ can also occur here and there. We assume the reader is acquainted with Fourier analysis of (tempered) distributions. In particular our most frequent symbols related to a distribution T on X will be $\operatorname{supp} T$ (= *the closed support of* T) and $\operatorname{spec} T = \operatorname{supp} \hat{T}$, the spectrum of T. Now we try to bring some order into the heap of "smallnesses" to be used below (see the statement of the UP above). We compose a list of properties of a function (or distribution) on X:

$S_1(f)$ ("fast bilateral decay of f at infinity"): f is defined on $\mathbb{R}$ or $\mathbb{Z}$ and satisfies

$$f(t) = O(M(t)), |t| \to +\infty$$

where M is a given majorant, $\lim_{|t|\to+\infty} M(t) = 0$:

$S_2(f)$: replacing $|t| \to +\infty$ in $S_1(f)$ by $t \to +\infty$ (or $t \to -\infty$) we get fast decay at $-\infty$ (or $+\infty$);

$S_3(f)$ ("a deep zero of f at a point $x_0 \in X$"): $f(t) = O(M(t)), t \to x_0$, where $\lim_{t\to x_0} M(t) = 0$;

$S_4(T)$ ("a sparse support") : $X \setminus \operatorname{supp} T$ is non-empty and more or less "rich" (say, consists of many long intervals or arcs),
but very interesting forms of the UP arise even when $X \setminus \operatorname{supp} T$ is connected, this is why the following three properties are stated separately:

$S_5(T)$ ("a gap in the support"): $\operatorname{supp} T$ omits a non-degenerate interval (or an arc) of X;

$S_6(T)$ ("a bounded support"): $\operatorname{supp} T$ is bounded (this case refers to $X = \mathbb{R}$);

$S_6(\hat{T})$ means that T is "band limited".

$S_7(T)$ ("a sembounded support"): $\operatorname{supp} T \subset [0, +\infty)$ (or $\subset (-\infty, 0]$); the spectral property $S_7(\hat{T})$ is absolutely fundamental for our subject and deserves special attention.

To conclude this list we have to define the so-called logarithmic integral $\mathcal{L}(f)$ of a function f defined on X:

$$\mathcal{L}(f) = \int\limits_{\mathbb{T}} \log|f|dm, \mathrm{if} X = \mathbb{T}; \mathcal{L}(f) = \int\limits_{\mathbb{R}} \log|f|d\Pi, \mathrm{if} X = \mathbb{R}$$

where Π is the Poisson measure $\pi^{-1}(1+x^2)^{-1}m$. Our last "smallness condition" means the geometric mean of f (w.r. to m on $\mathbb{T}$ or Π on $\mathbb{R}$) is zero; it looks less natural than its predecessors, but in fact it is *responsible* for many variants of the UP, and is omnipresent in many books on Fourier analysis, complex analysis and probability:

$$S_{\mathcal{L}}(f) : \mathcal{L}(f) = -\infty.$$

A discrete logarithmic integral $\mathcal{L}(f)$ of a function defined on $\mathbb{Z}$ will also play a role: this time $\mathcal{L}(f) = \sum\limits_{n \in \mathbb{Z}} \frac{\log|f(n)|}{1+n^2}$.

Now we are in a position to specify (slightly) our main question:

given j and $k = 1, 2, \ldots, 7$ or $\mathcal{L}$, is it true that $S_j(T)$ and $S_k(\hat{T})$ imply the complete vanishing of T?

The conditions S_j being still vague, the answer to such an "$(S_j, \hat{S}_k)$-question" depends heavily on the concrete relations between the time condition $S_j(f)$ and the frequency condition $S_k(\hat{f})$; some combinations of j and k may even not admit a satisfactory answer at all. But a remarkable fact is *the existence* of good answers to many such questions, the answers being sharp and verifiable. The list of the S_k-conditions looks dull (something like bookkeeping) and formal, but the diversity, variety and beauty of the ideas and tools required to answer at least some of our questions is quite amazing. Note that "qualitative" questions $(S_k, \hat{S}_j)$ usually entail some "quantitative" problems resulting in useful and explicit *estimates*.

The lecture consists of three parts. Part 1 is a collection of results not requiring any use of complex analyticity; we try to sketch (or at least allude to) some proofs (when they are simple). This policy becomes almost impossible in Part 2 based on complex analyticity. The "complex" proofs usually involve a good portion of hard analysis, so Part 2 is mainly a collection of results accompanied by some comments (kind of a guided tour). Part 3 is devoted to some remote repercussions of the UP: description of symbols of local and non-local shift invariant operators; some closely related topical problems stemming from potential theory are also discussed.

Our theme is present more or less explicitly in any course on Fourier analysis ([Z, Ba, Katz, Kah]), quasianalyticity ([M]), and complex analysis ([Bo, Du, Pr, Ho, Ga, Koo1]). The books [PW], [B], [L], [Lev], [KahS], [DeBr], [Koo2], [Koo3], [Car], [Carl], and [Nik] are especially close to our theme and have influenced our exposition in many ways.

The present lecture is mainly based on [HJ] just reproducing some of its parts in a very compressed form; we often refer to the bibliography therein. I also want to mention the long article [Na] with its impressive amount of excellent results.

1. The UP without Complex Variables

1.1. *On functions with semibounded spectra*

Let L be a linear shift invariant operator on the time line $\mathbb{R}$ defined on a vector space of functions (or distributions). The generic form of L is the convolution with a function (distribution) a:

$$Lf = a * f; \quad a = L(\delta).$$

We may interpret L as a device transforming inputs f into outputs Lf. We say L obeys the causality principle (or is *causal*) if $Lf|(-\infty, t_0)$ for any given moment t_0 depends only on $f|(-\infty, t_0)$ ("no output without an input"), or what is the same $a|(-\infty, 0) = 0$. In the Fourier coordinates the action of L becomes

$$\hat{Lf} = \hat{a} \cdot \hat{f}$$

which means Lf is a frequency filter. The causality imposes severe restrictions on the spectral characteristic $\hat{a}$ of L : $\hat{a}$ cannot suppress too many frequencies unless $L = 0$. This phenomenon stems from the analytic continuability of $\hat{a}$ into a half-plane (due to the semiboundedness of $\operatorname{supp} a$). This complex variable explanation will be one of the themes of Part 2. Actually one can explain many properties of $\hat{a}$ in the causal case staying on the line and ignoring the existence of the complex plane. One of these properties is *the Jensen inequality* for plus-functions (i.e. for functions with positive spectra).

It is convenient to interchange time and frequency lines and concentrate on the objects with semibounded *spectra* (rather than *supports*). We start with the periodic case: suppose $f \in L^p(\Pi, m) = L^p(\Pi),\ 1 \leqslant p \leqslant +\infty$, so that $\hat{f}$ lives on $Z : \hat{f}(n) = \int \bar{z}^n \, \mathrm{dm},\ n \in Z$. We say that f is in the Hardy class $H^p(\mathbb{T})$ if its spectrum is non-negative: $\operatorname{spec} f \subset Z_+$ (sometimes we call such an f a plus-function).

For a probability measure μ in a measure space X we put

$$\mathcal{A}_\mu(f)(=\mathcal{A}(f))=\int\limits_X |f|d\mu,\ \mathcal{G}_\mu(f)(=\mathcal{G}(f))=\exp\int\limits_X \log|f|d\mu,$$

thus defining *the arithmetic* and *geometric* means of f. Note that the meaning of the two "smallnesses"

$$\mathcal{A}(f)=0 \quad \text{and} \quad \mathcal{G}(f)=0(\Leftrightarrow \mathcal{L}(f)=\int\limits_X \log|f|d\mu=-\infty)$$

is very different: the first just means $f=0$ a.e. whereas the second is implied by $\mu(\{f=0\})>0$ or (depending on μ) by a fast decay of f at a point of X. By the Jensen inequality for the means we always have

$$\mathcal{G}(f)\leqslant\mathcal{A}(f). \tag{1}$$

Let us now go back to $X=\Pi,\ \mu=m$. Clearly, $|\hat{f}(0)|=|\int f dm|\leqslant\mathcal{A}(f)$ for any $f\in L^1(\Pi)$. But another Jensen inequality asserts that

$$|\hat{f}(0)|\leqslant\mathcal{G}(f), \quad \text{if} \quad f\in H^1(\mathbb{T}) \tag{2}$$

This crucial fact has many far-reaching implications pertaining to the UP. So, for example,

$$f\in H^1(\mathbb{T})\ \&\ \mathcal{G}(f)=0\Rightarrow f=0$$

($\mathcal{G}(f)=0$ kills $\hat{f}(0)$, but then it kills any $\hat{f}(n)$). But then

$$f\in H^1(\mathbb{T})\ \&\ |\{f=0\}|>0\Rightarrow f=0$$

(total absence of negative frequencies is not compatible with vanishing on a set of positive length). For a proof of (2) see, e.g., [HJ], p. 34; it is quite short and elementary.

By a Möbius change of variables we get the following version of (2): *If* $f\in L^1(\mathbb{R})$ *and* $\operatorname{spec} f\subset[0,+\infty)$, *then*

$$|\int\limits_{\mathbb{R}} f d\Pi|\leqslant\mathcal{G}_\Pi(f) \tag{3}$$

(f is regarded here as a tempered distribution, so $\hat{f}$ and $\operatorname{spec}\hat{f}$ make sense; Π is the Poisson measure, see the Introduction). In particular (3) is valid for any plus-function $f\in L^p(\mathbb{R})$ (w.r. to m), $1\leqslant p\leqslant+\infty$ (i.e. if $f\in H^p(\mathbb{R})=\{f\in L^p(\mathbb{R}):\operatorname{spec} f\subset[0,+\infty)\}$. It is easy to deduce from (3) that $f\in H^p(\mathbb{R})\ \&\ \mathcal{G}_\Pi(f)=0\Rightarrow f=0$ so that if $f\in H^p(\mathbb{R}),\ f\neq 0,$

then it cannot decay too fast at $+\infty$ or $-\infty$ or at any finite point, and it cannot vanish on a set of positive length. This last property is stable in the following sense:

THE THEOREM ON TWO CONSTANTS. *For any $f \in H^\infty(\mathbb{R})$ and any Lebesgue measurable $S \subset \mathbb{R}$*

(4) $$|(f * \Pi)(x)| \leqslant (\|f\|_{\infty,S})^{\Pi_x(S)}(\|f\|_{\infty,S'})^{\Pi_x(S')}, x \in \mathbb{R}.$$

$S' = \mathbb{R}\backslash S, \Pi_x(E) = \Pi(E-x))$.

The proof is a straightforward combination of two Jensen inequalities (1) and (3). Note that $\Pi_x(S)$ is the angle (divided by π) under which S is seen from $x+i$, and $f*\Pi_x = P(f)(x)$ is the Poisson integral of f (i.e. the bounded solution of the Dirichlet problem for the upper half-plane $\mathbb{C}_+$ with the boundary function f) computed at $x+i$. Clearly, $P(f) = 0$ implies $f = 0$; (4) shows that if a plus-function f is globally bounded (say, $|f| \leqslant 1$) and very small on S' (say, $|f| \leqslant \varepsilon$), then $P(f)$ is small globally: $|P(f)(x)| \leqslant \varepsilon^{\Pi_x(S)}$ for any real x.

We will also need an integral version of this result: *suppose $\gamma > 0$, and $\Pi_x(S) \geqslant \gamma$ for any $x \in \mathbb{R}$; if $f \in H^2(\mathbb{R})$, then*

(4′) $$\int\limits_{\mathbb{R}} |P(f)|^2 dm \leqslant 2(\int\limits_{S} |f|^2 dm)^\gamma \|f\|_2^{2(1-\gamma)}$$

where $\| \quad \|_2$ denotes the $L^2(m)$-norm ([HJ], p. 40).

The logarithmic integral $\mathcal{L}_\mu(f)$ figuring in $\mathcal{G}_\mu(f)$, $\mathcal{L}_\mu(f) = \int\limits_X \log|f|d\mu = \log \mathcal{G}_\mu(f)$ for $X = \Pi$, $\mu = m$ or $X = \mathbb{R}$, $\mu = \Pi$ plays an outstanding role in many problems concerning the UP (not only for semibounded spectra!). The two conditions

$$\mathcal{L}(f) = -\infty \quad \text{and} \quad \mathcal{L}(f) > -\infty$$

define two separate realms: in the first one the rule of the UP is indisputable whereas in the second it can be sometimes resisted (see [HJ], but especially [Koo2]).

1.2. *Hilbert Space methods*

1.2.1. *Annihilating pairs of sets.* For a function $f \in L^2(\mathbb{R}^d) = L^2$ the set $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ is called *the essential support of* f and denoted by $\operatorname{ess\,supp} f$; it is defined up to a set of zero Lebesgue measure. *The essential spectrum of* f is defined as $\operatorname{ess\,supp} \hat{f}$ and denoted by $\operatorname{ess\,spec} f$ ($\hat{f}$ is understood in accordance with the Plancherel theorem).

A pair (S, Σ) of Lebesgue measurable sets in $\mathbb{R}^d$ is said to be *annihilating* (or a-*pair*) if

(5) $$f \in L^2, \operatorname{ess\,supp} f \subset S, \operatorname{ess\,spec} f \subset \Sigma \Rightarrow f = 0.$$

The following property of (S,Σ) is more interesting: we say that (S,Σ) is *a strong* a-pair if

$$\int_{\mathbb{R}^d} |f|^2 \leqslant c(S,\Sigma)(\int_{S'} |f|^2 + \int_{\Sigma'} |\hat{f}|^2) \tag{6}$$

for any $f \in L^2$ (A' denotes $\mathbb{R}^d \setminus A$). The annihilation property (5) of a strong a-pair is “stable”: (5) only means that *vanishing* of $f|S'$ and $\hat{f}|\Sigma'$ implies *global vanishing* of f whereas (6) says that *the smallness* of $f|S', \hat{f}|\Sigma'$ implies *the global smallness of* f.

The d-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^d$ will be denoted by $|A|$.

The following version of the UP can be proved using only the basic properties of the Fourier transform and very general properties of projectors in a Hilbert space:

THE AMREIN-BERTHIER THEOREM.. *If*

$$|S| + |\Sigma| < +\infty,$$

then (S,Σ) *is a strong a-pair.*

Note that the sets $S, \sum$ are not supposed to be bounded. We are going to sketch a proof based on two orthogonal projectors $P_S, \hat{P}_\Sigma$ of L^2:

$$P_S f = \chi_S f, \mathcal{F}(\hat{P}_\Sigma f) = \chi_\Sigma \hat{f}$$

where χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^d$ and $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^d$ (duly normalized to define a unitary operator in L^2). The proof is sketched in 2.3 after some preparation in 2.2.

1.2.2. *Positive angle between two subspaces.* Let us now forget the concrete nature of these projectors and move to an abstract Hilbert space H; let (M,N) be a pair of its closed subspaces. We denote by P and Q the projectors of H onto M, N (resp.). We are interested in the following property of the pair (M,N) (or (P,Q)):

$$\|h\|^2 \leqslant c(M,N)(\|P^\perp h\|^2 + \|Q^\perp h\|^2) \text{ for any } h \in H, \tag{7}$$

where $P^\perp = I - P, Q^\perp = I - Q$ project onto the orthogonal complements of M and N (resp.). Clearly, (7) is an abstract form of (6). It can be given several equivalent forms:

(8)

(a) $\|PQ\|(= \|QP\|) < 1$;

(b) $\sup\{| < m,n > | : m \in M, n \in N, \|m\| \leqslant 1, \|n\| \leqslant 1\} < 1$;

(c) $M \cap N = \{0\}$, and $M + N$ is closed;

(d) $\|P^\perp n\| \geqslant c\|n\|$ for any $n \in N$ (or, equivalently, $\|Q^\perp n\| \geqslant c\|m\|$ for any $m \in M$), $c > 0$.

If M, N are of finite dimension, then all these properties just mean $M \cap N = \{0\}$, but in general this last property alone does not imply (8) which is often expressed as “the positivity of the angle between M and N " (look at (b)). A proof of the equivalence of (7) and all properties in (8) is, e.g., in [HJ], p. 80.

The following general observation is crucial for the Amrein-Berthier theorem: *If* $M \cap N = \{0\}$*, and* PQ *is compact, then* (7) *holds*. Indeed, PQ being compact there is a unit vector $v \in H$ such that $\|PQv\| = \|PQ\|$; if $\|PQ\| = 1$, then $1 = \|PQv\| \leqslant \|Qv\| \leqslant \|v\| = 1$ whence $\|Qv\| = \|v\|$ and $Qv = v$, so that $v \in N$ and $\|Pv\| = \|v\|, v \in M$ whereas the only element of $M \cap N$ is zero.

1.2.3. Let us now return to $L^2 = L^2(\mathbb{R}^d)$ and put $P = P_S, Q = \hat{P}_\Sigma, M = \{f \in L^2 : \operatorname{ess\,supp} f \subset S\}, N = \{f \in L^2 : \operatorname{ess\,spec} f \subset \Sigma\}$. If $|S|, |\Sigma|$ are finite, then PQ becomes an integral operator in L^2 with the kernel $H(x, y) = c\chi_S(x)\hat{\chi}_\Sigma(y-x)$ which is Hilbert-Schmidt: by Plancherel

$$\iint |H(x,y)|^2 dxdy = c^2 \int \chi_S^2 \cdot \int \chi_\Sigma^2 = c^2|S||\Sigma| < +\infty.$$

If $f \in M \cap N$ then f is an eigenvector of PQ corresponding to the eigenvalue 1, so $M \cap N$ is finite dimensional. Moreover, its dimension can be estimated by $|S||\Sigma|$:

$$\dim(M \cap N) \leqslant \iint |H(x,y)|^2 dxdy = c^2|S||\Sigma|. \tag{9}$$

Using this estimate it is not hard to prove that $M \cap N = \{0\}$ (i.e. that (S, Σ) is an a-pair), and thus complete the proof of the theorem. Suppose $\varphi \in M \cap N, \varphi \neq 0$, so that $S_0 \subset S, 0 < |S_0|$ where $S_0 = \operatorname{ess\,supp} \varphi$. For a vector $v \in \mathbb{R}^d$ put $\varphi_v(x) = \varphi(x - v)$. For a $v_1 \in \mathbb{R}^d$ the essential support S_1 of φ_{v_1} (i.e. $S_0 + v_1$) sticks out of S_0 (slightly):

$$0 < |S_1 \setminus S_0| < \varepsilon_1$$

where $\varepsilon_1 > 0$ is arbitrary (we are using the finiteness of $|S_0|$); functions φ, φ_{v_1} are linearly independent. Then we find $v_2 \in \mathbb{R}^d$ so as to make $S_2 = S_1 + v_1 = \operatorname{ess\,supp} \varphi_{v_1 v_2}$ stick out of $S_1 \cup S_0$ (slightly):

$$0 < |S_2 \setminus (S_1 \cup S_0)| < \varepsilon_2;$$

$\varphi, \varphi_{v_1}, \varphi_{v_1 v_2}$ are linearly independent. Continuing this process we arrive at an infinite linearly independent sequence of *shifts* of φ:

$$\varphi, \varphi_{v_1}, \varphi_{v_1 v_2}, \ldots \tag{10}$$

and the sequence of sets $S_0, S_0 + v_1, S_0 + v_1 + v_2, \ldots$ whose union S^* is of finite measure if only $\sum \varepsilon_j < +\infty$. The essential spectrum being shift invariant the sequence (10) is in $M^* \cap N, M^* = P_{S^*}(L^2), N = \hat{P}_\Sigma(L^2)$ which is impossible $(\dim(M^* \cap N) \leqslant c^2|S^*||\Sigma|)$.

1.2.4. *Some remarks on strong annihilation.* Using the equivalence of (7) and (8) and staying in an abstract Hilbert space H we may solve a series of problems pertaining to the UP. For example, if (8) holds, then the operator $v \to (Pv, Qv)$ maps H *onto* $M \times N$. This means that whenever (S, Σ) is a strong a-pair the following system of equations (with "the unknown" $r \in L^2$) is solvable:

$$r|S = p|S, \hat{r}|\Sigma = q|\Sigma$$

for any $p, q \in L^2$. Another example is the following problem: describe the image of the unit ball of H under the mapping $h \to (\|Ph\|, \|Qh\|) \in \mathbb{R}^2$. This problem can be solved quite explicitly for many pairs (P, Q) such that PQ is compact; the result is a quantitative refinement of the Amrein-Berthier theorem (the Slepian-Pollack inequality): roughly speaking the point $(\|Ph\|, \|Qh\|)$ of the square $[0,1] \times [0,1]$ cannot get too close to the vertex $(1,1)$. For a pair of sets S, Σ of finite measure the Slepian-Pollack inequality answers the following question: suppose $h \in L^2, \|h\| = 1$, and $\int\limits_S |h|^2 = \alpha$ with a given $\alpha \in (0,1)$; how large can $\int\limits_\Sigma |\hat{f}|^2$ be ? It turns out that the least upper bound of this "spectral energy carried by Σ" is one if $\alpha \leqslant c(S, \Sigma) < 1$, but it *does depend* on $\alpha \in (c(S, \Sigma), 1)$ remaining *strictly less* than one.

1.2.5. *The Paneah Theorem.* The definition of a strong a-pair suggests the following general question: given a class s of measurable sets $S \subset \mathbb{R}^d$ find the class

$$\hat{s} = \{\Sigma \subset \mathbb{R}^d : (S, \Sigma) \text{is a strong a-pair for any } S \in s\}.$$

Denoting by s_{fin} the class of all sets in $\mathbb{R}^d$ of finite Lebesgue measure we may restate the Amrein-Berthier theorem:

$$\hat{s}_{fin} \supset s_{fin}. \tag{11}$$

It is known that this inclusion is strict; I do not know any satisfactory and complete description of $\hat{s}_{fin}$. Let us turn instead to an important example of s when $\hat{s}$ admits a complete and explicit description: $s = s_b =$ the class of all *bounded* measurable sets in $\mathbb{R}^d$. We say that a Lebesgue measurable set $E \subset \mathbb{R}^d$ is *relatively dense* (at infinity) (or belongs to s_{rd}) if there exists a cube $K \subset \mathbb{R}^d$ and a number $\gamma > 0$ such that $|(K + x) \cap E| \geqslant \gamma$ for any $x \in \mathbb{R}^d$. A typical example (for $d = 1$) is the union of all intervals of a given positive length centered at equidistant points nh where $h > 0$ is fixed and $n \in \mathbb{Z}$. The rd-sets $E \subset \mathbb{R}$ can be *characterized* by the following property: the observer moving along the line $y = 1$ sees E all the time under an angle exceeding a positive number $\sum$; in other words

$$\Pi_x(E) = \frac{1}{\pi} \int\limits_E \frac{dt}{1 + (x - t)^2} \geqslant \sigma \text{ for any } x \in \mathbb{R}. \tag{12}$$

Denote by S'_{rd} the set of all complements of the rd-sets. The following theorem refers to $d = 1$ (i.e. to $\mathbb{R}$).

THE PANEAH THEOREM. $\hat{s}_b = s'_{rd}$.

(Paneah proved $\hat{s}_b \subset s'_{rd}$ for any dimension; the inverse inclusion in any dimension was proved by Logvinenko and Sereda later.) Here we sketch a short proof of a part of the Paneah theorem ([JH, Gor, HJ]), namely $s'_{rd} \subset \hat{s}_b$.

First note that the Poisson integral $P(\varphi) = \frac{1}{\pi}\varphi * \frac{1}{1+x^2}$ of $\varphi \in L^2$ is again in L^2, since

$$(\hat{P\varphi})(\xi) \equiv e^{-|\xi|}\hat{\varphi}(\xi) \tag{13}$$

(where $\hat{\varphi}(\xi) = (2\pi)^{-1} \int\limits_{\mathbb{R}} \varphi(x)\exp(-is\xi)dx$) whence

$$\|P(\varphi)\|_2 \leqslant \|\varphi\|_2.$$

If $\operatorname{spec}\varphi \subset [0,l]$, then an inverse estimate can be obtained:$|\xi|$ in (13) becomes ξ so that

$$|\hat{\varphi}(\xi)| = e^{\xi}|P\hat{(\varphi)}(\xi)| \leqslant e^{l}|P\hat{(\varphi)}(\xi)|,$$

and

$$\|\varphi\|_2 \leqslant e^{l}\|P(\varphi)\|_2 \tag{14}$$

by Plancherel. Suppose now $f \in L^2(\mathbb{R}), \operatorname{spec} f \subset [a,b]$; but then (14) is applicable to $\varphi = e^{-iax}f$ (with $l = b-a$). Applying the integral form of the two constants theorem to an rd-set E (see (4$'$) and (12)) we get

$$\|f\|_2^2 = \|\varphi\|_2^2 \leqslant e^{2l}\cdot 2\left(\int\limits_E |\varphi|^2\right)^{\Sigma} \|\varphi\|_2^{2(1-5)} = 2e^{2l}\left(\int\limits_E |f|^2\right)^{\Sigma} \|f\|_2^{2(1-5)}$$

whence $\|f\|_2^2 \leqslant 2e^{2l/\Sigma}\int_E |f|^2$. We have thus proved $(8d)$ for $P = P_S, S = E'$, and $Q = \hat{P}_{[a,b]}$ which means that $(S,[a,b])$ is a strong a-pair.

1.2.6. *Periodic case: strong annihilation of supports omitting a set of positive length and sparse spectra.* The problem setting of 1.2.1 has obvious $L^2(\mathbb{T})$-parallels, the corresponding definitions of a-pairs and strong a-pairs $(S,\Sigma), S \subset \mathbb{T}, \Sigma \subset \mathbb{Z}$ being essentially the same. E.g., (S,Σ) is a strong a-pair if

$$\int\limits_{\mathbb{T}} |f|^2 dm \leqslant c\left(\int\limits_{S'} |f|^2 dm + \sum_{n\in\Sigma'} |\hat{f}(n)|^2\right) \text{ for any} f \in L^2(\mathbb{T}).$$

This case can be also included into the general scheme of 1.2.2.

Let $s_{\mathbb{T}}$ be the class of all sets $S \subset \mathbb{T}$ satisfying $m(S) < 1$; denote $\hat{s}_{\mathbb{T}}$ by SPARSE. A deep and difficult result on SPARSE is due to Mikheev who proved that

$$\Lambda^*(2) \subset SPARSE \subset \Lambda(2)$$

where $\Lambda^*(2), \Lambda(2)$ are certain classes of rarefied sets of integers ($\Lambda(2)$ is very familiar to the specialists; see [HJ], p. 102-110 for definitions). Here we only mention that it is unknown whether $\Lambda^*(2) = \Lambda(2)$. It is known however that the finite unions of lacunary sets are in

$\Lambda^*(2)$ (a set A of positive integers is called *lacunary* if $\sup\{m/n : m, n \in A, m < n\} < 1$; a set A of integers is called lacunary if $A \cap (0, +\infty)$ and $\{|n| : n \in A, n < 0\}$ are lacunary). The strong annihilation of pairs (S, Σ) with $m(S) < 1$ and lacunary Σ (but not a finite union of lacunary sets) was proved by Zygmund.

1.3. *Review of some "non-complex" results*

The main source of concrete forms of the UP is, of course, Complex Analysis. This was avoided (or carefully masked) in the preceding parts of the lecture. Before turning to the powerful complex machinery I just want to mention some more forms of the UP susceptible to other methods.

1.3.1. *The Annrein-Berthier Theorem revisited.* To describe a new approach to this theorem we start with a proof (due to Benedicks) of an L^1-analog of annihilation of pairs $(S, \Sigma), S, \Sigma \subset \mathbb{R}, |S| + |\Sigma| < +\infty$: *if* $f \in L^1(\mathbb{R})(= L^1), f|S' = 0, \hat{f}|\Sigma' = 0$, *then* $f = 0$. The proof is based on two facts:

(i) If $\Sigma \subset \mathbb{R}, |\Sigma| < +\infty$, then for m-almost all $h \in (0, +\infty)$ almost all points of the lattice $(kh)_{k\in\mathbb{Z}}$ (i.e. all but a finite number) avoid Σ ([HJ], p. 456)

(ii) For $f \in L^1(\mathbb{R})$ put

$$p(t) = \sum_{k\in\mathbb{Z}} f(t+k)$$

the series converges in $L^1((-A, A))$ for any $A > 0$ thus defining a 1-periodic function p summable on $(0, 1)$, *the periodization of* f. Put $S = \operatorname{ess\,supp} f$, $\tilde{S} = \operatorname{ess\,supp} \cap (0, 1)$

it is easy to see that

$$m(\tilde{S}) \leqslant m(S). \tag{15}$$

The ε*-compression* f_ε of f is defined by $f_\varepsilon(x) = f(x/\varepsilon)$; by $p^{(\varepsilon)}$ we denote the periodization of f_ε. The k-th Fourier coefficient $p_k^{(\varepsilon)}$ of $p^{(\varepsilon)}$ (w.r. to the system $(\exp(2\pi ikx))$ is $\varepsilon\hat{f}(\varepsilon k)$. Hence, by (i), $p^{(\varepsilon)}$ is a trigonometric polynomial of period $1/\varepsilon$ for almost all ε provided $f|\sum' = 0, |\sum| < \infty$. So for any $f \in L^1(\mathbb{R})$ and $\varepsilon \to 0$

$$(p^{(\varepsilon)})_{1/\varepsilon} \to f \text{ in } L^1((-A, A))$$

([HJ], p. 458). Now we are ready to complete the proof: a sequence $(p^{(\varepsilon_k)})_{\frac{1}{\varepsilon_k}}$ of trigonometric polynomials (with $\varepsilon_k \to 0$) tends to f in L^1_{loc}; but if ε_k is small, then $p^{(\varepsilon_k)}$ vanishes on $(0, 1) \setminus \tilde{S}_{\varepsilon_k}$, a set of *positive length*, since $|\tilde{S}_{\varepsilon_k}| \leqslant \varepsilon_k|S|$ (by (15)) whence $p^{(\varepsilon_k)} \equiv 0$.

In the same spirit, but in a much more quantitative way Nazarov found an explicit estimate of the constant c in the Amrein-Berthier inequality (6). The abstract proof discussed in 1.2.2-1.2.3 did not yield any information on c; it was not even clear whether c depends on $|S|, |\Sigma|$ rather than on S, Σ.

THE NAZAROV THEOREM. *The Amrein-Berthier inequality (6) (for $d = 1$) holds with $c = A \exp A|S||\Sigma|$ where A is an absolute constant.*

Writing (6) for the Gauss function $f(x) = \exp(-x^2/2), S = \Sigma = [-N, N]$ we find $c \geqslant \exp A'|S||\Sigma|$ for an absolute A'. Note that in fact c tends to one as $|S||\Sigma| \to 0$ which fact can be easily deduced from the abstract geometric considerations of 1.2.2 and the estimate $\|P_S \hat{P}_{\sum}\|^2 \leqslant const|S||\Sigma|$; for $|S||\Sigma|$ bounded off zero the first factor A in Nazaror's estimate can be dropped.

An elementary probabilistic analysis of "random lattices" in the spirit of Benedicks argument led Nazarov to a "finite" version of the UP which is interesting in itself. In a particular case it was discovered by Turan in the fifties.

THE TURAN LEMMA. *Let P be a trigonometric polynomial*

$$(P(\xi) = \sum \hat{P}(n)\xi^n, \xi \in \mathbb{T}),$$

$\operatorname{spec} P$ *being a finite set of integers. Put ord* $P = \operatorname{card} \operatorname{spec} P$. *There exists an absolute constant C such that*

$$\max_{\mathbb{T}} |P| \leqslant (C/m(\gamma))^{ordP} \max_{\gamma} |P| \tag{16}$$

for an arbitrary arc $\gamma \subset \mathbb{T}$.

Note that $ordP \leqslant degP = \max\{|n| : n \in \operatorname{spec} P\}$, and (16) is an essentially non-linear result, since the set of all P's with a given $ordP$ is not a linear space. Turan's original proof was based on some explicit interpolation formulas. Nazarov needed (16) not for *arcs* γ, but for arbitrary compact subsets of $\mathbb{T}$, and he succeeded in proving (16) for this more general situation which required a new approach involving the Kolmogorov weak type estimate of the Hilbert transform. He proved along the way that if $\operatorname{spec} P \subset [-M, M]$, then for any $t > 0$

$$|\{\zeta \in \mathbb{T} : |P'(\zeta)| \geqslant tM|P(\zeta)|\}| \leqslant C_{abs}/t,$$

so that the Bernstein norm estimate of the derivative of a trigonomtric polynomial P of degree M ($\max_{\mathbb{T}} |P'| \leqslant M \max_{\mathbb{T}} |P|$) holds *pointwise* off a set of small measure ($\leqslant C_{abs}/t$) with tM in place of M.

1.3.2. *The F. and M. Riesz Theorem.* originally appeared and was perceived as a fact of Complex Analysis, but later it was given several proofs not using analytic functions. The theorem states that a charge (= a complex valued Borel measure) on $\mathbb{R}$ or $\mathbb{T}$ with positive spectrum (a plus-charge) is m-absolutely continuous. Of those non-complex proofs I mention here only one due to $A.B.$ Aleksandrov and J. Shapiro and based on peculiarities of the L^p-metric with $p \in (0, 1)$ restricted to trigonometrical polynomials $\sum_{n \geqslant 0} c_n z^n$ with non-negative spectrum. This approach is applicable to the charges on a multidimensioinal torus ([HJ], p. 41-50). An interesting quantitative version of the F. and M. Riesz theorem is due to Pigno, Smith ([HJ], p. 23-28).

1.3.3. *The De Leeuw-Katznelson Theorem.* The Fourier coefficients $\hat{\mu}(n)$ of a plus-charge on $\mathbb{T}$ tend to zero as $|n| \to +\infty$ (an immediate corollary of the F. and M. Riesz theorem). This property is stable: the De Leeuw-Katznelson theorem states that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any plus-charge μ on $\mathbb{T}$ with $var\mu \leqslant 1$

$$\limsup_{n \to -\infty} |\hat{\mu}(n)| < \delta \Rightarrow \limsup_{n \to +\infty} |\hat{\mu}(n)| < \varepsilon.$$

The proof is quite "real" ([HJ], p. 29-31).

1.3.4. *Spectral decay of singular charges.* An m-singular charge μ (on $\mathbb{T}$) is highly concentrated; the UP suggests its Fourier image $\hat{\mu}$ should be "spread", but to what extent? E.g., is it possible for $\hat{\mu}$ to tend to zero (in which case we call it an r-charge in honour of Rajchman)? The answer is "Yes." In this connection I mention a beautiful result due to Salem (preceded by a more elementary partial result due to Bari) characterizing the Cantor subsets of $\mathbb{T}$ whose Cantor measure is an r-measure ([HJ], p. 63, 86). Another device to produce singular r-measures are infinite Riesz products ([Z, HJ]; a very nice treatment of the Riesz products is in [Pey]).

An obvious spectral obstacle for a charge μ on $\mathbb{T}$ to be singular is the inclusion $\hat{\mu} \in l^2(\mathbb{Z})$. *The Ivashev-Musatov Theorem* asserts that this fact is sharp: for any "nice" non-negative function Φ defined on $[0, +\infty)$ with $\sum_1^\infty \Phi^2(n) = +\infty$ there exists a non-zero m-singular positive measure μ on $\mathbb{T}$ with compact support such that $|\tilde{\mu}(n)| \leqslant \Phi(|n|)$ for any $n \in \mathbb{Z}$ (we are not in a position to discuss here "the nicety" of Φ; dropping the compactness of $\operatorname{supp} \mu$ from the statement above we may just assume Φ to be decreasing ([Kor]). The proof is based on ingenious asymptotic estimates of oscillating integrals in the spirit of the Van-der-Corput lemmas.

1.3.5. *Deep zero & sparse spectrum.* Suppose $f \in C(\mathbb{T}), \varepsilon > 0, f(t) = O(\exp(-|t - 1|^{-(1+\varepsilon)}))$ as $t \to 1$ ("a deep zero at 1") and $\sum_{n \in \mathbb{Z}} |n|^{\varepsilon - 1/2} < +\infty$ ("a sparse spectrum"); then $f = 0$. This is a very particular case of the Mandelbrojt theorem. It was given a purely "real" proof by Belov ([HJ], p. 80-85). This proof is interesting in itself providing some useful quantitative relations. But the complex approach to the Mandelbrojt theorem results in its much stronger forms and seems to be the only way to prove its sharpness.

2. Complex Methods

2.1. *Introductory Remarks*

2.1.1. Partial sums of the Fourier series on $\mathbb{T}$ are rational functions, and partial Fourier integrals on $\mathbb{R}$ are entire functions; both live not only on $\mathbb{T}$ or $\mathbb{R}$, but in the whole ambient plane $\mathbb{C}$. Leaving $\mathbb{R}$ and $\mathbb{T}$ for $\mathbb{C}$ we get a vast new perspective making many manifestations of the UP just some uniqueness theorems of Complex Analysis. A (very primitive) example is this: if the support of a non-zero charge μ on $\mathbb{R}$ is bounded, then $\operatorname{spec}\mu$ is not, since $\hat{\mu}$ is an entire function. We can of course, strengthen this trivial remark replacing the boundedness of $\operatorname{supp}\mu$ by fast decay of μ at infinity (say, by the convergence of $\int\limits_{\mathbb{R}}(\exp c|t|)d|\mu|(t)$ for a $c > 0$ entailing the analyticity of $\hat{\mu}$ in the strip $\{|Imz| < c\}$. The wealth of subtle uniqueness theorems of Complex Analysis yields far more precise and profound forms of the UP.

The complex approach gives a new explanation of the UP phenomena for plus-charges on $\mathbb{R}$ and $\mathbb{T}$. Suppose a function f summable on the time axis $\mathbb{R}$ has no past, that is $f|(-\infty, 0) = 0$. Then its Fourier integral

$$\hat{f}(\xi) = (2\pi)^{-1}\int_{\mathbb{R}} f(t)\exp(-it\xi)dt$$

makes sense not only for *real* ξ, but also for any $\xi \in \mathbb{C}_-$, the open lower-half-plane, and f is *analytic there* (the condition $f \in L^1(\mathbb{R})$ is not essential, we might speak of a charge, an L^p-function, or a distribution living in the future, i.e. supported by $(0, +\infty)$). Reversing the order of time and frequency and changing sign in the exponent we conclude that any plus-function (distribution) f (i.e. when $\operatorname{spec} f \subset [0, +\infty)$) is in a way extendable to the upper half-plane $\mathbb{C}_+$ and this is (heuristically) a *complete* characterization of the plus-functions: if f is extendable from $\mathbb{R}$ to a function analytic in $\mathbb{C}_+$ satisfying some growth conditions, then f is a plus-function (this is not a theorem, but rather a useful heuristic principle).

An analogous description of the plus-functions on $\mathbb{T}$ is even more obvious. A Fourier series $\sum\limits_{n\geqslant 0}\hat{f}(n)z^n$ lacking negative harmonics becomes a power series converging in the open unit disc $\mathbb{D}$, and the interpretation of a plus-function f on $\mathbb{T}$ as a boundary trace of its sum looks very plausible (and, similarly $\sum\limits_{n<0}\hat{f}(n)z^n$ becomes a Laurent series in $\mathbb{C}\setminus(\mathbb{D}\cup\mathbb{T})$).

These remarks are meaningful and rich in consequences for *arbitrary* functions (not just for plus- or minus-functions): a more or less arbitrary function f on $\mathbb{R}$ can be written as $\int\limits_{-\infty}^{+\infty}\hat{f}(\xi)e^{it\xi}d\xi$ (in a sense) whence

$$f = f_+ - f_- \text{where} f_+(t) = \int\limits_0^{+\infty}\hat{f}(\xi)e^{it\xi}d\xi, f_-(t) = -\int\limits_{-\infty}^{0}\hat{f}(\xi)e^{it\xi}d\xi, \tag{17}$$

so that $f_\pm$ are plus- and minus-functions extendable to the respective half-planes. An analogous decomposition is valid for functions (and distributions) on $\mathbb{T}$: if, say, $f \in L^1(\mathbb{T})$, then putting $f_+ = \sum\limits_{n\geqslant 0}\hat{f}(n)z^n, f_- = -\sum\limits_{n<0}\hat{f}(n)z^n$ we get $f = f_+ - f_-$, a formal equality to be duly interpreted which is quite possible in many cases.

Let us remember the following form of the UP: a continuous non-zero plus-function on $\mathbb{T}$ cannot vanish on a set of positive length (see section 1.1 of Part 1). This fact becomes now a boundary uniqueness theorem for functions analytic in $\mathbb{D}$ and continuous up to $\mathbb{T}$, and a conformal mapping of $\mathbb{C}_+$ onto $\mathbb{D}$ immediately yields an analogous result for the plus-functions on the line.

2.1.2. The complex point of view gives a new insight into the notions of *support* and *spectrum*. The support of a (say, summable) function f on $\mathbb{R}$ can be characterized as the set of singularities of its Cauchy potential Φ,

$$\Phi(\xi) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)dt}{t-\xi}, \quad (\xi \in \mathbb{C} \setminus \mathbb{R})$$

since $f(t) = \lim\limits_{\varepsilon \downarrow 0}(\Phi(t+i\varepsilon) - \Phi(t-i\varepsilon))$ a.e. on $\mathbb{R}$. But $\operatorname{spec} f = \operatorname{supp} \hat{f}$, and (if $\hat{f}$ is summable and in many other cases) $\operatorname{spec} f$ is the set of singularities of the function Φ, analytic in $\mathbb{C} \setminus \mathbb{R}$ defined as $\Phi(\zeta) = (2\pi i)^{-1} \int\limits_{\mathbb{R}} (\hat{f}(\xi)/(\xi - \zeta))d\xi$ which is readily seen to be $-(2\pi)^{-1} \int\limits_0^{+\infty} f(t)\exp(-it\zeta)dt$ for $Im\zeta < 0$ and $(2\pi)^{-1} \int\limits_0^{+\infty} f(t)\exp(-it\zeta)dt$ for $Im\zeta > 0$. A similar description of $\operatorname{supp} f$ for $f \in L^1(\mathbb{T})$ is obvious: it coincides with the set of singularities of Φ: $\zeta \to (2\pi i)^{-1}\int_{\mathbb{T}} f(t)(t-\zeta)^{-1}dt(|\zeta| \neq 1)$, or the complement of the largest open set $O \subset \mathbb{T}$ such that Φ is analytic in $\mathbb{D} \cup O \cup \{|\zeta| > 1\}$.

2.1.3. Versions of the UP obtained by the complex tools are very often based on the following fact: *suppose* $F \neq 0$ *is analytic in a domain* $O \subset \mathbb{C}$*; then* $\log|F|$ *is subharmonic in* O*:*

$$\log|F(a)| \leqslant \int\limits_{\mathbb{T}} \log|F(a+rz)|dm(z) \tag{18}$$

provided the disc $\{|z-a| \leqslant r\}$ *is in* O (this is Jensen inequality (2)). The subharmonicity is akin to convexity, and (18) implies a certain rigidity of $|F|$: the smallness of $|F|$ on a small (but solid) part P of O makes $|F|$ small on $O \setminus P$ as well. A rigorous statement of this kind is the two constants inequality (4). The subharmonicity of $\log|F|$ for an analytic F entails the following extremely useful heuristic principle: *if a non-zero analytic function is not too big (globally), then it cannot be too small (even locally).* If for example $\operatorname{supp} f \subset [-\sigma, \sigma], f \in L^1([-\sigma,\sigma])$, then $|\hat{f}(\zeta)| \leqslant \mathit{const}\exp(\sigma|\zeta|), \zeta \in \mathbb{C}$, which is a global growth restriction imposed onto entire function $\hat{f}$; an appropriate form of the Jensen inequality forbids $\hat{f}$ to decay too fast along $\mathbb{R}$ or to have too many zeros.

To conclude these introductory remarks let me mention the very special plasticity of *the formulas* of Complex Analysis. So, for example, a band limited function has at least four faces: it is a trigonometric integral $\int\limits_{-\sigma}^{\sigma} \hat{f}(\xi)\exp(i\xi t)d\xi$, but also a power series, or a contour integral, the Borel transform of the entire function $\hat{f}$ (with a freedom to deform the contour), or an infinite canonical product.

Now we start our guided tour around (some) applications of the complex tools to the UP.

2.2. *Fast decay of f and $\hat{f}$ at infinity*

A pair (M, N) of positive functions on $(0, \infty)$ is called *sufficient* if

$$|f(t)| \leqslant M(|t|) \& |\hat{f}(\xi)| \leqslant N(|\xi|)(t, \xi \in \mathbb{R}) \Rightarrow f = 0. \tag{19}$$

A complete description of sufficient pairs $(e^{-At^p}, e^{-B\xi^r})$ is due to Morgan who proved in 1934 that such a pair is sufficient if $1/p + 1/r < 1$ (not depending on $A, B > 0$) and found the conditions to be imposed on (A, B) to make the pair sufficient for given p, r satisfying $1/p + 1/r = 1$. His main tool was the Phragmen-Lindelöf theorem (a far reaching generalization of the maximum modulus principle). Using another complex tool (the Carleman formula for a contour integral involving the logarithm of a function analytic in $\mathbb{C}_+$) Dzhrbashyan obtained some sufficiency criteria (replacing the pointwise majorization (19) by integral estimates); he also got a description of some "unilaterally sufficient" pairs (that is those (M, N) for which $f = 0$ follows from the inequalities in (19) if both (or one) of them are fulfilled only on the ray $(0, +\infty)$). The following elegant result is due to Beurling: if $f \in L^1(\mathbb{R})$, and $\iint_{\mathbb{R}\times\mathbb{R}} |f(t)||\hat{f}(\xi)|e^{|t||\xi|} dt d\xi < +\infty$, then $f = 0$ ([B]). The results of this section (with their proofs and references) can be found in [HJ], p.128-137; we also recommend Nazarov's article [Na] containing a new approach to sufficient pairs.

2.3. *Deep zero & fast decay at infinity*

Let H be a non-negative function on $[0, +\infty)$. In this section we call it *sufficient* if

$$f(t) \cdot t^n = O(1)\ (|t| \to \infty, n \in \mathbb{Z}_+)\ \&\ |\hat{f}(t)| \leqslant H(|t|)\ (t \in \mathbb{R}) \Rightarrow f = 0.$$

If $H(t)t^n = O(1)$ $(t \to +\infty)$ for any $n > 0$, then "the depth of zero" of f at the origin just means $f^{(n)}(0) = 0, n \in \mathbb{Z}_+$. This is actually a classical quasianalyticity problem related to the moment problem and weighted polynomial approximation.

A *necessary* condition for H to be sufficient is $\mathcal{L}(H) = -\infty$ ($\mathcal{L}(H) = \int_0^{+\infty} \log H d\Pi$, see Part 1 section 1.1); this "quantitative" condition is *sufficient* if H satisfies a "qualitative" *regularity condition* (which cannot be dropped), e.g., if H is logarithmically convex $(H(x) = \exp(-x/\log(x+1))$ or $\exp(-x/\log(x+1)\log\log(x+1)\ldots)$ are sufficient (see [Koo2], [HJ]). This combination of a quantitative condition (19) and a qualitative one (the regularity of the majorant) is very typical for many forms of the UP.

2.4. *Deep zero & sparse spectrum*

Here we return to the Mandelbrojt theorem (see 1.3.5 of Part 1) and briefly discuss its proof due to Levin (this proof results actually in a much stronger theorem which we won't state here, see [L], [HJ]). The strategy of the proof is this: suppose $f \in L^1(\mathbb{T})$ has a deep zero at 1 (say, $\int\limits_0^{\varepsilon} |f(e^{it})| dt = O(e^{-\varepsilon^{-\rho}})$ as $\varepsilon \downarrow 0$; ρ is positive); look at the Laplace transform

Φ of the periodic function $\varphi, \varphi(t) = f(e^{it})(t \in \mathbb{R}); \Phi(p) = \int\limits_0^{+\infty} \varphi(t)e^{-pt}dt$ coincides in $\{Rep > 0\}$ with the meromorphic function $\sum_{n\in\mathbb{Z}} \hat{f}(n)(p-in)^{-1}$ of a very tempered growth off the union of the discs $\{|z-in| < 1/4\}$; the deep zero of f at 1 (that is the deep zero of φ at the origin) can be translated as the fast decay of $|\Phi(\xi + i\eta)|$ as $\xi \uparrow +\infty$ (uniformly in η). The Poisson-Jensen formula

$$\int\limits_0^R (N(t)/t)dt = \int\limits_{\mathbb{T}} \log|\Phi(R\zeta)|dm(\zeta) - \log|\Phi(0)|$$

where $N(t) = n(t) - p(t), n(t), p(t)$ being, resp., the numbers of zeros and poles in $t\mathbb{D}$, implies the estimate

$$\int\limits_0^R (p(t)/t)dt \geqslant -\left(\int_{\mathbb{T}\cap\{Re\zeta>0\}} + \int_{\mathbb{T}\cap\{Re\zeta<0\}}\right)\log|\Phi(R\zeta)|dm(\zeta) + \log|\Phi(0)|;$$

the first integral tends to $-\infty$ and so does the whole bracket, since the modulus of the second integral grows too slowly; if Φ is regular at the origin and $\Phi(0) \neq 0$ (which we may assume if $\varphi \neq 0$), then $p(t)/t$ has to be big for arbitrarily large values of t, and an excessive sparseness of $\operatorname{spec} f$ (= the set of poles of Φ) is impossible. We can now go back: given a sufficiently sparse set Λ of integers we construct a meromorphic function $\Phi = 1/B$ where B is a suitable infinite product (an entire function) vanishing exactly at the points in, $n \in \Lambda$, and growing fast enough off the discs $\{|z-in| < a\}$ for a positive a; applying the Riemann-Mellin inversion formula for the Laplace transform to Φ we obtain a periodic φ and then $f \in C(\mathbb{T})$ with $\operatorname{spec} f = \Lambda$ and a deep zero at 1; this is how the sharpness of the Mandelbrojt theorem is proved.

2.5. *Semibounded spectra*

The complex point of view sheds a new light upon the functions with semibounded spectra. Here we can only mention the highly developed theory of the Hardy classes H^p whose main objects are L^p-functions with non-negative frequencies.

Let f be an L^p-function $(1 \leqslant p \leqslant +\infty)$ on $X = \mathbb{T}$ or $\mathbb{R}$ with a non-negative spectrum: $\operatorname{spec} f \subset [0, +\infty)$ (if $p > 2, X = \mathbb{R}$, then $\operatorname{spec} f$ is the support of the distribution $\hat{f}$); then we say $f \in H^p(X)$, the Hardy space on X. Another object related to $H^p(X)$ is a Banach space $H^p(O)$ of functions analytic in $O(= \mathbb{D}$ for $X = \mathbb{T}, \mathbb{C}_+$ for $X = \mathbb{R})$ satisfying a certain L^p-growth restriction. It turns out that any $F \in H^P(O)$ has finite boundary values along the normals m -a.e. on X thus defining a function $F^* \in H^p(X)$,

$$F^*(t) = \lim_{r\uparrow 1} F(rt) \quad (t \in \mathbb{T}), F^*(t) = \lim_{\varepsilon\downarrow 0} F(t+i\varepsilon) \quad (t \in \mathbb{R}).$$

The mapping $F \to F^*$ takes $H^P(O)$ isometrically onto $H^p(X)$, so that the $L^p(X)$-functions with no negative frequencies can be *identified* with the *analytic* functions in $H^p(O)$. This close connection makes it possible to understand *completely* many forms of the UP for the plus-functions (including continuous and smooth plus-functions, see [Du], [Pr], [Ho], [Ga], [Koo1], [HJ]; these forms are usually sharp in contrast with other spectral "smallnesses" (e.g., for the band limited functions, see section 2.6 below). As an example we consider here a complete and quite satisfactory description of the moduli of the $H^p(X)$-functions.

THEOREM. *Let $h \geqslant 0$ be a non-zero function on X. The following are equivalent: (i) $h = |f|$ where $f \in H^p(X)$; (ii) $h \in L^p(X, m)$ and $\mathcal{L}(h) > \infty$ (see section 1.1 in Part 1).*

Thus the convergence of the logarithmic integral $\mathcal{L}(h)$ is the *only* smallness restriction for the equation $|f| = h, f \in H^p(X)$ to be solvable. Its necessity follows immediately from the Jensen inequality; its sufficiency can be proved by an explicit construction: if $h \in L^p(\mathbb{T}, m)$ and $\mathcal{L}(h) > -\infty$, then $\operatorname{Ext} h : z \mapsto \exp \int\limits_{\mathbb{T}} \log h(\zeta) \frac{\zeta+z}{\zeta-z} dm(\zeta)$ is in $H^p(\mathbb{D})$ and $|(\operatorname{Ext} h)^*| = h$ a.e. on $\mathbb{T}$; $\operatorname{Ext} h$ is the so-called *outer* (or *exterior*) function corresponding to h. An analogous formula can be written for $X = \mathbb{R}$ as well.

The conditions of the solvability of the equation $|f| = h$ with an unknown band limited function f (i.e. with a *bounded* and not just semibounded spectrum) can hardly be expressed in palpable terms (see however [Dy] for some useful results in this direction). In the next section in place of the equation $|f| = h$ we turn to non-trivial band limited solutions of *the inequality* $|f| \leqslant h$.

2.6. *Fast decay at infinity and bounded spectrum*

Let h be a non-negative function defined on $\mathbb{R}$. We call it *a Beurling-Malliavin majorant* (BM-*majorant*) if there exists a non-zero function f with a bounded spectrum such that

(20) $$|f| \leqslant h.$$

A bounded set being semibounded we immediately conclude that any BM-majorant h satisfies $\mathcal{L}(h) > -\infty$. But in contrast with section 2.5 this condition is far from being sufficient: to guarantee the solvability of (20) with a band limited $f \neq 0$ we have to impose some *regularity* conditions on h to moderate its oscillations at infinity. The reason is simple: a band limited function is entire and of finite degree (i.e. $f(t) = O(\exp \sigma|t|), t \in \mathbb{C}, |t| \to +\infty$): the Poisson-Jensen formula shows that the zeros of f tend to run away with a certain speed from any (big) disc (so that the number of zeros of f in $r\mathbb{D}$ is $O(r)$ as $r \uparrow +\infty$). And if, say, $h(\sqrt{n}) = 0, n = 1, 2, \ldots$, or even if $h(\sqrt{n})$ tends to zero fast enough, then (20) implies $f = 0$; but this behavior of h is very well compatible with $\mathcal{L}(h) > -\infty$ if only the slopes of the pits on the graph of h over $\sqrt{n}$ are steep, i.e. if h oscillates intensely. (There are, of course, even much more obvious obstacles for an h with $\mathcal{L}(h) > -\infty$ to be a BM-majorant: for example $h(t) = \exp(-1/\sqrt{|t|})$ is not a BM-majorant. But we concentrate now on BM-majorants bounded off zero on any bounded interval.)

Assume $\mathcal{L}(h) > -\infty$; the characterization of the oscillations of h at infinity compatible with h being a BM-majorant is a very hard problem. A remarkable breakthrough is due to Beurling and Malliavin. Their work [BM1] describing a large class of BM-majorants is deep and difficult involving a good deal of potential theory and complex analysis. Here I state only one corollary: *Suppose h is bounded and strictly positive; if $\mathcal{L}(h) > -\infty$ and $\log h$ satisfies a Lipschitz condition, then h is a BM-majorant.*

Another (and even more famous) corollary is the so-called Beurling-Malliavin multiplier theorem which I won't state here. Subsequent proofs, simplifications, and approaches to these corollaries are due to Koosis, Kargaev, and Nazarov see [Koo2, Koo3]; one more proof is in [DeBr]. But the original result still seems to remain the most general (it is also exposed in [HJ], p. 306-369).

Note that if h is even and decreasing on $[0, +\infty)$ (no oscillations at all), then $\mathcal{L}(h) > -\infty$ is sufficient for h to be a BM-majorant. This fact is relatively simple and known actually for a long time before the Beurling-Malliavin theorem had been proved (see the references and a proof in [HJ], p. 276).

Another "whale", the second Beurling-Malliavin theorem, can be only named here. It is devoted to the following form of the UP: bounded support & missing frequencies (characterization of the discrete sets $\Lambda \subset \mathbb{R}$ such that $\hat{f}|\Lambda = 0$ for a function $f \neq 0$ concentrated on $[-\sigma, \sigma], \sigma > 0$; see [BM2, Koo2, HJ]). Denoting by χ_A the characteristic function of a set A we may rephrase the problem: describe the BM-majorants of the form $\chi_{\mathbb{R}\setminus\Lambda}$.

2.7. *Four theorems on the unilateral decay*

This series of theorems starts historically with the following result due to Levinson and Cartwright:

(I) Suppose $f \in L^1(\mathbb{T})$ satisfies

$$|\hat{f}(n)| \leqslant h(|n|) \text{ for all negative integers } n, \tag{21}$$

$h : [1, +\infty) \to (0, +\infty)$ being a decreasing function ("unilateral decay of $\hat{f}$"). If

$$\sum_{n=1}^{\infty} \frac{\log h(n)}{n^2} = -\infty, \tag{22}$$

then f cannot vanish identically on a non-degenerate arc unless $f = 0$.

In 1960 Beurling proved the following result which is in fact much stronger than (I): for a finite charge μ on $\mathbb{R}$ and $A > 0$ put $\rho_\mu(A) = (var\mu)([A, +\infty))$;

(II) if

$$\int_1^{+\infty} \frac{\log \rho_\mu(A)}{A^2} dA = -\infty, \tag{23}$$

then $\hat{\mu}$ cannot vanish identically on a set of positive length unless $\mu = 0$.

This time "the unilateral decay" refers to "the object" μ and the spectral smallness means vanishing on a large set. The remarkable feature of this result is the total absence of the regularity conditions (cf. section 2.3 and 2.6). The strategy of the proof is killing the Cauchy potential $C(\mu)(\zeta) = \int\limits_{\mathbb{R}} (t-\zeta)^{-1} d\mu(t) (\zeta \notin \mathbb{R})$ for any μ satisfying (23) with $\hat{\mu}$ vanishing on a set of positive length. The Levinson-Carwright theorem (I) follows from (II) very easily.

(III) (The Volberg Theorem) Suppose the conditions of (I) are fulfilled and h satisfies some supplementary regularity conditions (not to be stated here); then $\mathcal{L}(f) > -\infty$ unless $f = 0$.

The conclusion of this theorem is much stronger than in (I), but it does not imply (I) because of those unnamed regularity conditions (their sharp form is due to J. Brennan, see [Koo2, HJ]); the regularity of h in (I) is its mere decrease. Theorem (III) was conjectured by Dyn'kin in 1975; its proof is based on Dyn'kin's theory of pseudoanalytic continuation and delicate estimates of pseudoanalytic functions in the so-called boundary layers.

(IV) (The Borichev Theorem) The last result of this series is due to Borichev and looks (at first glance) even stronger than (III). It is applicable not only to *functions* on $\mathbb{T}$, but to *distributions* and even to *hyperfunctions*. Any two-sided sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers such that $\limsup\limits_{|n|\to\infty} |a_n|^{1/|n|} \leqslant 1$ generates two analytic functions g_+, g_-:

$$g_+(\zeta) = \sum_{n\geqslant 0} a_n \zeta^n (|\zeta| < 1), g_-(\zeta) = -\sum_{n<0} a_n \zeta^n (|\zeta| > 1).$$

If $|a_n| = O(|n|^m)$ for a positive m, then $\sum\limits_{n\in\mathbb{Z}} a_n\zeta^n$ is the Fourier series of a distribution T on $\mathbb{T}$ and $\operatorname{supp} T$ is the complement of the largest open part of $\mathbb{T}$ across which g_+ is analytically extendable to $-g_-$. The Borichev theorem asserts in particular that if

$$\lim_{n\to-\infty} \log|a_n|/h(|n|) = -\infty, \limsup_{n\to+\infty} \log|a_n|/h(n) < +\infty,$$

and h satisfies (22) and some regularity conditions (again !) then it is impossible for the non-tangential limits of g_+ and g_- to coincide on a subset of $\mathbb{T}$ of positive length unless $a_n \equiv 0$. In fact, [Bor, BorV] contain much stronger results involving the divergence of a logarithmic integral (to be defined properly, since $\sum a_n\zeta^n$ is not a function on $\mathbb{T}$, and $g_\pm$ are not bound to possess boundary values on $\mathbb{T}$).

For the most popular classical majorants

$$h(n) = \exp(-cn/\log n \cdot \log\log n \cdot \ldots)$$

each of these four theorems is a step forward compared with the preceding one. But in general none of them implies the rest (because of the discrepancies in the regularity conditions). Their proofs are different,and it is unclear whether they can be encompassed by a single statement and proof.

2.8. *Spectral gap & sparse support*

We say a tempered distribution T on $\mathbb{R}$ has a *spectral gap* if $\mathbb{R} \setminus \operatorname{spec} T$ contains a non-degenerate interval. The Beurling theorem ((II) in section 2.7 above) forbids the fast unilateral decay of a function (or a charge) with a spectral gap. But now we are going to discuss *the sparseness of the support* of a charge with a spectral gap. One more Beurling theorem gives an answer: *Suppose $S \subset \mathbb{R}$ is a closed set such that*

$$\int\limits_{\mathbb{R}} \frac{dist(x,S)}{1+x^2} dx = +\infty.$$

Then any non-zero charge μ supported by S has no spectral gap ([B, Koo2, HJ]).

A proof due to Koosis is based on the Pollard approach to weighted approximation and the Bernstein band limited function $\cos \sigma \sqrt{(z-x_0)^2 - R^2}$ peaking at $x_0 \in \mathbb{R}$ and bounded by one off $(x_0 - R, x_0 + R)$. This result is only an illustration. The problem to characterize the support carrying a charge with a spectral gap has impressive connections with potential theory,weighted approximation by polynomials and entire functions of finite degree (see the results by Levin, Akhiezer & Levin, Kargayev, Benedicks, Koosis,De Branges, Levin & Logvinenko & Sodin quoted in [HJ], p.375). Here we only mention that the Beurling theorem of this section is sharp in the following sense: the spectral gap cannot be replaced in its statement by a set of positive length; this was predicted by Koosis and proved by Kargaev's counterexample (his original construction was simplified by Kislyakov and Nazarov, see [HJ], p.520).

We conclude this section by the following problem posed by Sapogov: is there a set $A \subset \mathbb{R}$ of finite length whose characteristic function χ_A has a spectral gap? The answer is yes, it is due to Kargayev who constructed such an A as the union of disjoint intervals I_n gravitating to n as $|n| \to +\infty$; their endpoints can be computed by the Newton-Kantorovich method in $l^2(\mathbb{Z})$ which yields alot of information on I_n (see [HJ], p.376-392 and the paper by Kargaev & Volberg quoted there).

2.9. *Sparse support & unilateral decay*

A compact set $K \subset \mathbb{T}$ is called *spacious* if it carries a non-zero charge μ such that

$$|\hat{\mu}(n)| = O(|n|^{-m}) \; (n \to -\infty) \tag{24}$$

for any $m > 0$. The characterization of "bilaterally spacious" sets K (i.e. carrying a charge $\mu \neq 0$ satisfying (24) with $|n| \to \infty$) is easy: they are just the sets with interior points. The unilateral character of (24) makes the description of spacious sets a much more delicate task. It is obvious that the length of a spacious K is positive, since any μ satisfying (24) is m-absolutely continuous by the F. and M. Riesz theorem. But this condition is not sufficient.

Denote by $\mathcal{L}(K)$ the set of all components of $\mathbb{T} \setminus K$ and call

$$\sum_{l\in\mathcal{L}(K)} |l| \log |l|$$

the entropy of K.

THE HRUŠČEV THEOREM. *K is spacious iff it contains a compact subset of positive length and finite entropy (see* [Hru, HJ]*).*

This is a difficult result. It uses among other things some variants of the Khinchin-Ostrowski theorem on normal families of functions analytic in a disc, delicate estimates of outer functions and a clever construction of a special measure on $\mathbb{T}$ (its simplified version due to N. Makarov is in [HJ]). The Hruščev theorem stated above is just a representative of a long series of his results on “sparse supports & unilateral decay” including many concrete sorts of “decay” and various “objects” (not necessarily charges); only a part of the results of [Hru] is in [HJ].

3. Local and Antilocal Convolutions

This part is devoted to a form of the UP for shift invariant linear operators; we call it *antilocality*. The most interesting examples and problems come from potential theory and are discussed in section 3.3. A class of antilocal operators is the theme of section 3.2, in section 3.1 we discuss *local* (“almost differential”) operators as opposed to the antilocality of sections 3.2 and 3.3. In this part everything is closely related to the UP of Parts 1 and 2.

3.1. *Local and completely local convolutions*

We denote by $\mathcal{D}'(\mathbb{R}^d)$ the set of all distributions in $\mathbb{R}^d$. Let $\mathcal{K}$ be a linear operator mapping a linear set $dom\mathcal{K} \subset \mathcal{D}'(\mathbb{R}^d)$ into $\mathcal{D}'(\mathbb{R}^d)$. We call it *local* if it does not increase the support:

$$T \in dom\mathcal{K} \Rightarrow \operatorname{supp}\mathcal{K}T \subset \operatorname{supp} T,$$

or what is the same $\mathcal{K}(T)|O$ depends only on $T|O$ for any open $O \subset \mathbb{R}^d$ and $T \in dom\mathcal{K}$. A typical example is any linear differential operator with C^∞-coefficients; this is in a sense the only possible example: it can be proved under some mild conditions to be imposed on $dom\mathcal{K}$ (but not in general !) that local operators are differential (the Peetre theorem).

A local operator $\mathcal{K}$ reproduces any *open* zero set E of a distribution:

(25) $$T \in dom\mathcal{K}, T|E = 0 \Rightarrow \mathcal{K}T|E = 0.$$

Suppose $dom\mathcal{K}$ and $im\mathcal{K} = \mathcal{K}(dom\mathcal{K})$ consist of *locally summable functions* so that $T|E$ and $\mathcal{K}T|E$ make sense for $T \in dom\mathcal{K}$ and any Lebesque measurable set $E \subset \mathbb{R}^d$. Then we call $\mathcal{K}$ *completely local* if (25) holds for any such E (not only open). Any linear differential operator whose domain consists of sufficiently smooth functions is completely local.

Now we turn to the shift invariant operators $\mathcal{K}$ on $\mathbb{R}$: Let K be a Lebesque measurable function on $\mathbb{R}$. Consider the convolution operator $\mathcal{K}$

(26) $$dom\mathcal{K} = \{f \in L^2 : K\hat{f} \in L^2\}, \hat{\mathcal{K}f} = \widehat{Kf} \ (f \in dom\mathcal{K}).$$

De Branges found a complete characterization of the symbols of local operators.

THE DE BRANGES THEOREM. *(a) Suppose K is a restriction to $\mathbb{R}$ of an entire function k of the Cartwright class and degree zero (that is $k(\zeta) = O(\exp \varepsilon|\zeta|), |\zeta| \to +\infty$, for any $\varepsilon > 0$, and $\mathcal{L}(k) < +\infty$); then $\mathcal{K}$ is local;*

(b) suppose $dom\mathcal{K}$ contains a non-zero function vanishing on a non-degenerate interval; if $\mathcal{K}$ is local,then $K = k|\mathbb{R}$ for an entire function k of the Cartwright class and degree zero.

In other words local shift invariant operators with a sufficiently rich domain are precisely "the almost differential linear operators with constant coefficients"; they can be written formally as $t \to \sum\limits_{k=0}^{\infty} a_k t^k$ where $\sum\limits_{k=0}^{\infty} a_k \zeta^k$ represents a slowly growing entire function, not too far from a polynomial.This result is only a particular case of a much more precise theorem due to De Branges and describing the symbols of the so-called σ-local convolutions ([DeBr, HJ]).

In the extreme case of a polynomial symbol K our operator $\mathcal{K}$ becomes a usual linear differential operator with constant coefficients defined on the Sobolev space $W_2^n = \{f \in L^2 : f \in C^{(n-1)}, f^{(n-1)}$ absolutely continuous, $f^{(n)} \in L^2, n = degK\}$. In this case $\mathcal{K}$ is not just *local*, but *completely local*. De Branges posed the following question: suppose $\mathcal{K}$ is *almost* differential (as in his theorem above); is it completely local ? A counterexample was constructed by Kargayev. He actually showed that $f \to \sum\limits_{k=0}^{\infty} a_k f^{(k)}$ can be not completely local even for an entire symbol $k(\zeta) = \sum\limits_{k=0}^{\infty} a_k \zeta^k$ of *order* zero (i.e. $k(\zeta) = O(\exp|\zeta|^{\varepsilon})$ for any $\varepsilon > 0$), see [HJ],p.482-484.

3.2. *Complete antilocality*

The results of section 3.1 are in fact closely related to the themes of Part 2, but only with respect to the tools. We turn now to an opposite property of certain convolutions (the antilocality) which is *itself* a form of the UP for a shift invariant operator forbidding us to know too much on the operator. Our theme here is the compulsory increase of the support of a non-zero function under certain convolution operators.

A linear operator $\mathcal{K}$ defined on $dom\mathcal{K} \subset \mathcal{D}'(\mathbb{R}^d)$ with values in $\mathcal{D}'(\mathbb{R}^d)$ is called *antilocal* if

$$T \in dom\mathcal{K}, T \neq 0 \Rightarrow \operatorname{supp} \mathcal{K}(T) \supset \mathbb{R}^d \setminus \operatorname{supp} T$$

which means that the following UP holds: for any non-empty open $E \subset \mathbb{R}^d$ and $T \in dom\mathcal{K}$

(27) $$T|E = \mathcal{K}(T)|E = 0 \Rightarrow T = 0,$$

forbidding the simultaneous vanishing of T and $\mathcal{K}(T)$ on a solid (=open) set. This UP is valid for some interesting convolution operators and is akin to its "harmonic" prototype. The simplest example is the Hilbert transform (in $\mathbb{R}$) taking $f \in L^2$ to $\tilde{f}, \tilde{f}(x) = p.v. \int\limits_{\mathbb{R}} f(t)/(t - x)dt$; in spectral terms this means $\hat{\tilde{f}}(\xi) = c \cdot sgn\xi \cdot \hat{f}(\xi)$. If $f, \tilde{f}$ both vanish on an open $E \subset \mathbb{R}$, then the function $\varphi : \zeta \mapsto \int\limits_{\mathbb{R}} f(t)/(t - \zeta)dt$ analytic in the domain $\mathbb{C}_+ \cup E \cup \mathbb{C}_-$ vanishes on E and is thus identically zero; being the jump of φ as its argument crosses $\mathbb{R}$, $f = 0$ a.e. on $\mathbb{R}$. It is not hard to see that the logarithmic potential $f \mapsto f * \log|x|$ and the M.Riesz potentials $f \mapsto f * |x|^{-\alpha}$ $(0 < \alpha < 1)$ are antilocal. But the Hilbert transform and the logarithmic potential enjoy in fact a much stronger uniqueness property: they are *completely* antilocal. We say that $\mathcal{K}$ is *completely antilocal* if (27) holds not only for any open E, but for any set E of positive Lebesgue measure in $\mathbb{R}^d$ (we assume that $dom\mathcal{K}$ and $im\mathcal{K}$ consist of locally summable functions). It is sometimes quite hard to prove (or disprove) that an antilocal operator is completely antilocal; for the Hilbert transform this property coincides with the UP for $H^2(\mathbb{R})$, see section 1.1 of Part 1.

The symbols of local operators described by the De Branges theorem of section 3.1 consist of a single analytic block being polynomials or entire functions. It turns out that many symbols consisting of *two different* "analytic blocks" define an antilocal (or even a completely antilocal) operator.

Let K be a Lebesgue measurable function on $\mathbb{R}$; $b, c \in \mathbb{R}, b < c$. Suppose K coincides on $(c, +\infty)$ with a rational function r, and $|\{\xi : \xi \leqslant b, r(\xi) = K(\xi)\}| = 0$. Then we call K a *semirational symbol* and r its rational part. A typical example is $K(\xi) = sgn\xi \cdot r(\xi)$ where r is rational (if $r \equiv 1$,then we get the Hilbert transform).

Define $\mathcal{K}$ by (26). This operator is completely antilocal for many semirational symbols, e.g., for any $K(\xi)$ of the form $sgn\xi/q(\xi)$ where q is a polynomial. But for general semirational symbols (27) is only proved under a supplementary smoothness conditions to be imposed on T ; (27) can be restored for *all* $T \in dom\mathcal{K}$ provided E satisfies an extra "entropy condition" as in the Hruščev theorem (section 2.9 of Part 2), and it is unknown whether these supplementary conditions can be dropped ([HJ], p.484-488 and the references therein including papers by Havin, Joericke, Makarov,and Ch.Bishop). For example it is unknown whether (27) is true for $K(\xi) = sgn\xi \cdot (\xi - i)/(\xi - 2i)$ (it is true if we assume $T \in W_2^1$).

Another interesting antilocal convolution is the M. Riesz potential

$$(\mathcal{K}(f) = f * |x|^{-\alpha}, \alpha \in \mathbb{R}, \alpha \neq -2, -4, \ldots)$$

whose symbol $c|\xi|^{\alpha-1}$ also consists of two *different* analytic pieces. As to the *complete* antilocality, it is only known that for $\alpha \in (0,1)$ the following property holds: *if*

$$\int\limits_{|t|>1} |f(t)||t|^{-\alpha}dt < +\infty, \tag{28}$$

$$E \subset \mathbb{R}, |E| > 0, f|E = (f * |x|^{-\alpha})|E = 0, \tag{29}$$

and f satisfies a Hölder condition (depending on α) near E,then $f = 0$ (see [HJ], p.499-508 where all real values of α are also considered). Thus an extra smoothness condition emerges here once again although the proof is quite different from the proof of the UP for semirational symbols. But in this case these conditions cannot be dispensed with.Using a method due to Bourgain & Wolff a non-zero continuous function f on $\mathbb{R}$ has been constructed in [BH] which satisfies (28) and (29).

3.3. *A uniqueness problem for the Newton potentials*

An extremely interesting example of the antilocal behaviour can be observed on the Newton convolution in $\mathbb{R}^2$, i.e. on the operator

$$f \to U^f,\ U^f(x) = \int\limits_{\mathbb{R}^2} \frac{f(y)dy}{|x-y|}, x \in \mathbb{R}^2$$

(or, more generally, on the M.Riesz potentials

$$U^f_\alpha(x) = \int\limits_{\mathbb{R}^d} \frac{f(y)dy}{|x-y|^{d-\alpha}}, \alpha \neq 2, 4, \ldots, x \in \mathbb{R}^d).$$

The antilocality of the Newton potential U can be interpreted as a uniqueness property of the solutions of the Cauchy problem for the Laplace equation in the upper half-space $\mathbb{R}^3_+$, and it is very close to a boundary uniqueness property of harmonic (=divergence- and curl-free) vector fields in $\mathbb{R}^3_+$, a three dimensional analog of the boundary uniqueness theorem for functions analytic in the upper half-plane $\mathbb{C}_+$ (= a UP for plus-functions, see section 1.1 of Part 1 and section 2.5 of Part 2).

A remarkable construction due to Bourgain and Wolff [BW] (preceded by a breakthrough in [W] and some simplifications due to Aleksandrov and Kargayev [AK]) has disproved the complete antilocality of the Newton potential in $\mathbb{R}^2$: there exists a continuous non-zero function f in $\mathbb{R}^2$ such that $U^f = f = 0$ on a set of positive area in $\mathbb{R}^2$. It is, however, unknown whether such an example is possible with a smooth (say C^1, not to mention C^∞) function f. The one-dimensional construction of [BH] related to the M.Riesz potentials suggests that the answer may be negative. The antilocality properties of the Newton potentials are one of the themes of [HJ] (see p.488-508, and the references to the papers of Mergelyan, Landis, M.M.Lavrentjev, N.Rao, and Maz'ya & Havin).

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