Some Unsolved Problems in Harmonic Analysis as found in Analytic Number Theory

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1. *Suppose that* $T(x) = \sum_{n=0}^{N} c_n \cos n^2 x$, *that* $T(0) = 1$, *and that* $T(x) \ge 0$ *for all* x . *Among all such trigonometric polynomials, let* $\delta(N)$ *denote the minimum possible value of* c_0 . *It is known, via indirect reasoning, that* $\delta(N) \to 0$ *as* $N \to \infty$. *How fast, as a function of* N , *does* $\delta(N)$ *tend to* 0*? Like* N^{-a}*? Like* $(\log N)^{-a}$*?* ...

2. **(Fürstenberg)** *Let* p *and* q *be positive integers, not powers of the same number. Suppose that* x *is a real number, and let* $\mathcal{S}_p$ *be the set of fractional parts of the numbers* $p^n x$. *Similarly, let* $\mathcal{S}_q$ *be the set of fractional parts of the numbers* $q^n x$. *Show that if* x *is irrational then the sum of the Hausdorff dimensions of the closures of these sets is* ≥ 1 .

From this it would follow that 2^n has a 7 in its base 10 expansion, for all large n .

3. *Let* $\|\theta\|$ *denote the distance from* θ *to the nearest integer. From a classical theorem of Dirichlet we know that for any real* θ *and any positive integer* N *there is an* n , $1 \le n \le N$, *such that* $\|n\theta\| \le 1/(N+1)$. *We want something similar, but using only squares: For every real* θ *and every positive integer* N *there is an* n , $1 \le n \le N$, *such that* $\|n^2\theta\| \le f(N)$, *with* $f(N)$ *as small as possible.*

Heilbronn showed that one may take $f(N) = N^{-1/2+\varepsilon}$, and more recently A. Zaharescu [16] showed that this can be improved to $f(N) = N^{-4/7+\varepsilon}$. Can this be improved to $f(N) = N^{-1+\varepsilon}$?

4. *Show that if* $P(x) = \sum_{j=1}^{k} \alpha_j x^j$, $|\alpha_k - a/q| \le 1/q^2$, $(a, q) = 1$, *then*

$$\sum_{n=1}^{N} e\big(P(n)\big) \ll_k N^{1+\varepsilon}\Big(\frac{1}{q} + \frac{q}{N^k}\Big)^{1/k}.$$

Alternatively, construct examples that violate this.

Even small improvements of existing bounds would be interesting. For example when $k = 3$ and $q \approx N^{3/2}$, derive an upper bound that is $o(N^{3/4})$.

5. *Show that for any positive number* B *there exist complex numbers* $z_1, \ldots, z_N$ *such that* $|z_n| = 1$ *for all* n *, and* $\sum_{n=1}^N z_n^\nu \ll_B N^{1/2}$ *uniformly for* $1 \le \nu \le N^B$ *.*

This is known when $B = 2$.

6. *Suppose that* $|z_n| \ge 1$ *and that* $b_n \ge 0$ *for all* n *. Prove that*

$$\max_{1 \le \nu \le 2N} \Big| \sum_{n=1}^N b_n z_n^\nu \Big| \gg \Big(\sum_{n=1}^N |b_n|^2 \Big)^{1/2},$$

or give a counter-example.

This was proved by Gonek [5] under the stronger hypothesis that $|z_n| = 1$ for all n .

7. *Suppose that* N *points* $\boldsymbol{u}_n$ *are given in* $\mathbb{T}$ *. Let* $\mathcal{B}$ *be a box with one corner at* $\mathbf{0}$ *and the diagonally opposite corner at* $\boldsymbol{\alpha}$ *, and let* $D(\boldsymbol{\alpha}) = \sum_{n=1}^N \chi_{\mathcal{B}}(\boldsymbol{u}_n) - N \,\mathrm{area}(\mathcal{B})$ *denote the discrepancy of the* $\boldsymbol{u}_n$ *with respect to this box. Finally, let* D *denote the supremum of* $|D(\boldsymbol{\alpha})|$ *over all boxes. It is known that points* $\boldsymbol{u}_n$ *can be chosen so that* $|D(\boldsymbol{\alpha})|$ *is uniformly bounded by* $c_k(\log N)^{k-1}$ *. Is it always the case that* $\|D\|_\infty$ *is at least this order of magnitude?*

This is known to be the case when $k = 2$, first proved by W. M. Schmidt [14] in 1971, and then by Halász [6] later by a modified form of Roth's method. But it is not known for larger k.

8. *Let* N *points be given in* $\mathbb{T}^k$ *, and let* $D(\boldsymbol{\alpha})$ *denote the associated discrepancy function, as defined above. Show that* $\|D\|_1 > c_k(\log N)^{(k-1)/2}$ *.*

This was proved by Halász [6] when $k = 2$, but the case $k > 2$ is open.

9. *Show that if* $|a_n| \le 1$ *for all* n *, then*

$$\int_0^T \Big| \sum_{n=1}^N a_n n^{-it} \Big|^{2p} \, dt \ll (T + N^p) N^{p+\varepsilon}$$

uniformly for $1 \le p \le 2$ *.*

This is easy to prove when $p = 1$ or $p = 2$. The Density Hypothesis concerning zeros of the Riemann zeta function follows from the above.

10. *Suppose that* $0 \le t_1 < t_2 < \ldots < t_R \le T$ *with* $t_{r+1} - t_r \ge 1$ *for all* r *, and* $|a_n| \le 1$ *for all* n *. Show that*

$$\sum_{r=1}^R \Big| \sum_{n=1}^N a_n n^{-it_r} \Big|^2 \ll (N + R) N^{1+\varepsilon}.$$

11. *Let* $\lambda_1, \ldots, \lambda_N$ *be distinct real numbers, and put* $\delta_n = \min_{\substack{m \\ m \ne n}} |\lambda_m - \lambda_n|$ *. Determine the best constant* c *in the inequality*

$$\sum_{\substack{m,n\\m\neq n}} \frac{x_m \overline{x_n}}{\lambda_m - \lambda_n} \le c \sum_{n=1}^{N} \frac{|x_n|^2}{\delta_n} .$$

Certainly the best constant is no smaller than π . Montgomery and Vaughan [11] proved the above inequality with $c = 3\pi/2$.

12. **(Littlewood)** *Let $\|\theta\|$ denote the distance from θ to the nearest integer. Is it true that*

$$\liminf_{n\to\infty} n\,\|n\theta\|\,\|n\phi\| = 0$$

for any real numbers θ and ϕ?

13. **(Lehmer)** *The Mahler measure of a polynomial $P(x) = a_n x^n + \cdots + a_0 = a_n(x-\alpha_1)\cdots(x-\alpha_n)$ is the quantity*

$$M(P) = |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|) .$$

Show that there is a constant $c > 1$ such that if $P \in \mathbb{Z}[x]$ and $M(P) < c$ then P is a product of cyclotomic polynomials (i.e. $M(P) = 1$) .

The least known value of $M(P) > 1$ was found by Lehmer [9] long ago, in an exhaustive search. Lehmer's polynomial $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ has arisen many times since. It achieves $M(P) = 1.1762808$.

Let $\theta > 1$ satisfy $\theta^3 - \theta - 1 = 0$. Siegel proved that θ is the least PV number. Smyth [15] proved that if P is irreducible and $M(P) < \theta$ then P is a reciprocal polynomial, that is, $P(x) = x^n P(1/x)$. More recent numerical studies have been conducted by Boyd [2].

14. **(Salem)** *An algebraic integer $\alpha > 1$ is called a Salem number if all conjugates of α lie in the closed unit disk $|z| \le 1$, with at least one conjugate on the unit circle. Let $\mathcal{S}$ denote the set of all PV numbers, and $\mathcal{T}$ the set of all Salem numbers. Salem proved that every member of $\mathcal{S}$ is a limit of members of $\mathcal{T}$. Does $\mathcal{T}$ have any other limit points? Prove that 1 is not a limit point of $\mathcal{T}$. Prove that $\mathcal{S} \cup \mathcal{T}$ is closed.*

For basic properties of PV and Salem numbers, see Salem [13] or Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse and Schreiber [1]. See also numerous papers of Boyd, and papers cited therein.

15. *Show that there is a constant $c > 0$ such that if an algebraic integer α and all its conjugates lie in the disk $|z| \le 1 + c/n$ then α is a root of unity.*

The best result in this direction thus far is due to Dobrowolski [4], who proved that there is a positive constant c such that if

$$M(\alpha) < 1 + c\left(\frac{\log\log n}{\log n}\right)^3$$

then α is a root of unity. The value of c has been improved, and the proof simplified, by Cantor and Straus [3], Rausch [12] and Louboutin [10].

16. **(D. J. Newman)** *Let* $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$ *be independent random variables with* $P(\varepsilon_n = 0) = P(\varepsilon_n = 1) = 1/2$ *, and put* $K(z) = \sum_{n=0}^{N} \varepsilon_n z^n$ *. Show that there is an absolute constant* $c > 0$ *such that*

$$P\Big(\min_{|z|=1} |K(z)| < 1\Big) > c\,.$$

Does there exist a choice of the $\varepsilon_n \in \{0,1\}$ *such that* $\min_{|z|=1} |K(z)| > c$ *? What if* c *is replaced by* $c\sqrt{N}$ *?*

17. **(Hardy–Littlewood)** *Suppose that* $f \in L^2(\mathbb{T})$ *, and that* $\widehat{f}(k) \ne 0$ *only when* k *is a perfect square. Does it follow that* $f \in L^p(\mathbb{T})$ *for all* $p < 4$ *?*

A. Córdoba has shown that the answer is in the affirmative if one assumes that the numbers $\widehat{f}(k^2)$ are positive and monotonically decreasing.

18. *Consider a simple closed curve in the plane. A point* P *is called equichordal if every line through* P *meets the curve at two points that are a constant distance apart. Can a curve have two distinct equichordal points?*

19. **(Erdős)** *Let* $\mathcal{A}$ *be an infinite set of positive integers, and let* $r(n)$ *be the number of ways of writing* n *as a sum of two members of* $\mathcal{A}$ *. If* $r(n) > 0$ *for all* n *, does it follow that* $\limsup_{n\to\infty} r(n) = +\infty$ *?*

20. **(Veech)** *Suppose that* α *is given,* $0 < \alpha < 1/2$ *. Does there exist a function* f*, not identically* 0*, such that* f *has period* 1*, is even,* $f(0) = 0$ *,* $\sum_{a=1}^{q} f(a/q) = 0$ *for every positive integer* q*, and* $f \in \mathrm{Lip}(\alpha)$ *?*

21. *For* $P \in \mathbb{C}[z]$ *let* $\|P\|$ *denote the* ℓ_1 *norm of the coefficients of* P *. Suppose that* A *and* B *are polynomials of degrees* a *and* b *, respectively. It is known that the ratio* $\|A\|\,\|B\|/\|AB\|$ *always lies between* 1 *and* 2^{a+b} *provided that* $AB \ne 0$ *. What is the 'usual' or 'average' size of this ratio, as one averages over polynomials in a suitable way?*

22. **(H. Maier)** *Let* $\Phi_n(z)$ *denote the* n^{th} *cyclotomic polynomial,*

$$\Phi_n(z) = \prod_{\substack{a=1\\(a,n)=1}}^{n} (z - e^{2\pi i a/n}) = \sum_{m=0}^{\phi(n)} c(m,n) z^m\,,$$

Put $C(n) = \max_m |c(m,n)|$ *. Show that* $(\log C(n))/\log n$ *has a limiting distribution as* $n \to \infty$ *.*

23. **(Schinzel)** *Let* $\Phi_n(z)$ *denote the* n^{th} *cyclotomic polynomial, as above. Show that if* n *is squarefree then* Φ_n *has* $\gg n^{1/2}$ *non-zero coefficients.*

24. **(Balazard)** *Does there exist a Dirichlet series* $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *that converges in a half-plane* $\Re s > \alpha$ *with the property that the equation* $D(s) = 0$ *has exactly one solution in this half-plane?*

If the Riemann Hypothesis is true then $1/\zeta(s)$ is such Dirichlet series, since then one can take $\alpha = 1/2$ and the only zero is at $s = 1$. The object is to give an unconditional construction.

25. **(Schinzel)** *Suppose that* $f(x) = \sum_{i=0}^{n} a_i x^i$ *with* $a_i \ge 0$ *for all* i, *and that* $f(x)^2 = \sum_{i=0}^{2n} b_i x^i$ *with* $b_i \le 1$ *for all* i. *Let* $\mathcal{S}(n)$ *denote the set of all such polynomials. It is known that* $\max_{f\in\mathcal{S}(n)} f(1)^2 \sim Cn$. *Show that* $C = 4/\pi$.

Schmidt proved that $4/\pi \le C \le 2$ and Schinzel showed that $C \le 7/4$.

26. **(Ruzsa)** *Let* $\mathcal{A}$ *be a set of positive integers with positive asymptotic density. It follows that gaps between members of the set* $\mathcal{A} - \mathcal{A}$ *are bounded. It does not follow that gaps between members of the set* $\mathcal{A} + \mathcal{A}$ *are bounded. Does it follow that gaps between members of the set* $\mathcal{A} + \mathcal{A} + \mathcal{A}$ *are bounded?*

27. **(Melfi–Erdős)** *Let* $\mathcal{A} = \left\{ \sum \epsilon_i 3^i : \epsilon_i = 0, 1 \right\}$, *and let* $\mathcal{B} = \left\{ \sum \epsilon_i 4^i : \epsilon_i = 0, 1 \right\}$. *Does the set* $\mathcal{A} + \mathcal{B}$ *have positive lower density?*

28. **(Brüdern)** *Let* $\mathcal{P}$ *denote the set of prime numbers. Suppose that a set* $\mathcal{A}$ *is chosen so that* $\mathcal{A}+\mathcal{P}$ *has density* 1. *Let* $A(N)$ *denote the number of members of* $\mathcal{A}$ *not exceeding* N. *How slowly can* $A(N)$ *grow? Certainly one can have* $A(N) < (\log N)^C$. *What is the least possible value of* C?

29. **(Skinner)** *Let* f *be a non-constant polynomial with rational coefficients such that* $f(i)$ *is* 0 *or* 1 *for* $i = 0, 1, \ldots, n$. *How small can the degree of* f *be?*

Clearly $[n/2]$ is a lower bound, and n is an upper bound.

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