The Problem of Efficient Inversions and Bezout Equations

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ABSTRACT. This is a survey of some recent results on the phenomenon of the "invisible spectrum" for Banach algebras. Function algebras, formal power series and operator algebras are considered. This includes a quantitative treatment of the famous Wiener-Pitt-Sreider phenomenon for convolution measure algebras on locally compact abelian (LCA) groups. Efficient sharp estimates for resolvents and solutions of higher Bezout equations in terms of their spectral bounds are considered. The smallest spectral "efficiency hull" of a given closed set is introduced and studied. Using the spectral hulls we define uniformly bounded functional calculi for elements of the algebras in question. This program is realized for the measure algebras of LCA groups and for the measure algebras of a large class of topological abelian semigroups; for their subalgebras—the (semi)group algebra of LCA (semi)groups, the algebra of almost periodic functions, the algebra of absolutely convergent Dirichlet series, as well as for the weighted Beurling-Sobolev algebras, for H^∞ quotient algebras, and for some finite dimensional algebras.

Part I. Quantitative version of the Gelfand theory

1. Introduction: Efficient Inversion

1.1. *Basic motivations*

There are three classical problems of harmonic analysis and function theory related to a phenomenon we call *invisible spectrum.* Formal definitions are contained in Subsection 1.2 below.

The *first problem* comes from convolution equations and is related to what is usually called *"the Wiener-Pitt phenomenon".* Namely, let G be a locally compact abelian group (*LCA* group) written additively, and $\mathcal{M}(G)$ the convolution algebra of all complex measures on G. The fundamental problem is to find an invertibility criterion for measures $\mu \in \mathcal{M}(G)$, that is, a criterion for the existence of $\nu \in \mathcal{M}(G)$ such that $\mu * \nu = \delta_0$, δ_0 being the unit of $\mathcal{M}(G)$ (the Dirac δ-measure at 0). An obvious obstruction for invertibility is the vanishing of the Fourier transform $\hat{\mu}(\gamma) = 0$ at a point γ of the dual group $\hat{G}$, since the equality $\hat{\mu}\hat{\nu} \equiv 1$ is equivalent to the initial convolution equation. Generally, the boundedness away from zero

$$\delta = \inf_{\gamma \in \widehat{G}} |\widehat{\mu}(\gamma)| > 0 \tag{1.1}$$

is necessary for μ to be invertible. N. Wiener and R. Pitt (1938), and Yu. Sreider (1950, a corrected version of Wiener and Pitt's result) discovered that, in general, this is not sufficient. Namely, there exists a measure μ on the line $\mathbb{R}$ whose Fourier transform

$$\hat{\mu}(y) = \int_{\mathbb{R}} e^{-ixy}\, d\mu(x), \quad y \in \mathbb{R} \tag{1.2}$$

is bounded away from zero but there exists no measure $\nu \in \mathcal{M}(\mathbb{R})$ such that $\hat{\nu}(y) = 1/\hat{\mu}(y)$ for $y \in \mathbb{R}$. This result still holds true for an arbitrary *LCA* group G which is not discrete, see [Ru1], [GRS], [HR] for references and historical remarks. Using the Banach algebra language, one can say that the dual group $\hat{G}$, being the "visible part" of the maximal ideal space $\mathfrak{M} = \mathfrak{M}(A)$ of the algebra $A = \mathcal{M}(\mathbb{T})$, is far from being a dense subset of $\mathfrak{M}$. Nonetheless, later on we will see that some quantitative precisions of (1.1), namely a closeness of the norm $\|\mu\|$ and δ of (1.1), lead to the desired invertibility, and even to a norm control of the inverse.

On the contrary, for a discrete group G, the classical Wiener theorem on absolutely convergent Fourier series tells that condition (1.1) implies the invertibility of μ, and, moreover, $\mathfrak{M} = \widetilde{G}$. However, in this setting too, the problem of the norm control for inverses μ^{-1} is still meaningful and interesting, in spite of the Wiener theorem, since the latter does not yield any estimate. In fact, from the quantitative point of view, there is no big difference between these qualitatively polar cases (we mean the cases of nondiscrete and discrete groups). It turns out that, in both cases, one can control the norms $\|\mu^{-1}\|$ for $\delta > 0$ close enough to the norm $\|\mu\|$, but this is not the case for small $\delta > 0$.

The *second problem* we are interested in is to distinguish, among all unital Banach algebras A, those permitting an estimate of the resolvents only in terms of the distance to the spectrum. More precisely, we want to know for which algebras A there exists a function φ such that

$$\left\|(\lambda e - f)^{-1}\right\| \leqslant \varphi(\operatorname{dist}(\lambda, \sigma(f))) \tag{1.3}$$

for all $\lambda \in \mathbb{C}\backslash\sigma(f)$ and all $f \in A$, $\|f\| \leqslant 1$. Here e stands for the unit of A, and $\sigma(f)$ for the spectrum of f in the algebra A. We treat this problem in a more general context of norm-controlled functional calculi, that is, as a partial case of the norm estimates problem for functions operating on a Banach algebra. See Chapter 2 for more details.

The *third problem* is a multi-element version of the previous two. Postponing precise definitions and discussions to Subsection 1.2, we mention here the classical corona problem for the algebra $H^\infty(\Omega)$ of all bounded holomorphic functions on Ω, an open subset of $\mathbb{C}^n$ or of a complex manifold. Recall that the problem is to solve the Bezout equations

$$\sum_{k=1}^{n} g_k f_k = 1 \tag{1.4}$$

in the algebra $H^\infty(\Omega)$, where the data $f_k \in H^\infty(\Omega)$ satisfy an analogue of condition (1.1),

$$\delta^2 = \inf_{z\in\Omega} \sum_{k=1}^{n} |f_k(z)|^2 > 0, \tag{1.5}$$

and to estimate solutions $g_k \in H^\infty(\Omega)$. The Banach algebra meaning of the corona problem is well known; namely, the existence of $H^\infty(\Omega)$ solutions for any data satisfying (1.5) is equivalent to the density of Ω, the "visible part" of the spectrum $\mathfrak{M} = \mathfrak{M}(H^\infty(\Omega))$, in $\mathfrak{M}$. In what follows, we consider a norm refinement of this problem for several algebras different from $H^\infty(\Omega)$.

1.2. *(In)Visibility levels. Main problems*

Let A be a commutative unital Banach algebra and X be a subset of the maximal ideal space of A (the spectrum of A), $\mathfrak{M} = \mathfrak{M}(A)$, $X \subset \mathfrak{M}(A)$. We write $f \longmapsto \hat{f}(\mathfrak{m})$ for $\mathfrak{m} \in M$, or simply $f \longmapsto f(\mathfrak{m})$, for the Gelfand transform of an element $f \in A$, and hence, staying on X, we can embed A into $C(X)$, $f \longmapsto f(x) = \delta_x(f)$ for $x \in X$, where $\delta_x \in \mathfrak{M}$ stands for the point evaluation $\delta_x(f) = f(x)$, $f \in A$. In what follows, we regard $\operatorname{clos} X$ as the *visible part* of $\mathfrak{M}$.

Recall that the spectrum $\sigma(f)$ of an element $f \in A$ coincides with the range $f(\mathfrak{M})$ of the Gelfand transform. The following definition formalizes different levels of "visibility" of the spectrum.

DEFINITION 1.2.1. *The spectrum of A is called n-visible* (or, *n-visible from X*), $n = 1, 2, \dots$, if $f(\mathfrak{M}) = \operatorname{clos}(f(X))$ for all n-tuples $f = (f_1, \dots, f_n) \in A^n = A \times \dots \times A$; and it is called *completely visible* if $\mathfrak{M} = \operatorname{clos} X$.

It is clear that $(n+1)$-visibility implies n-visibilty for any $n \geqslant 1$, and the complete visibility is equivalent to n-visibility for all $n \geqslant 1$. Moreover, the Gelfand theory of maximal ideals makes evident the following lemma.

LEMMA 1.2.2. *For a commutative unital Banach algebra A, the following properties are equivalent.*

(i) The spectrum of A is n-visible.

(ii) For every $f = (f_1, \dots, f_n) \in A^n$ satisfying

$$\delta^2 =: \inf_{x \in X} \sum_{k=1}^{n} |f_k(x)|^2 > 0, \tag{1.6}$$

there exists an n-tuple $g \in A^n$ solving the Bezout equation

$$\sum_{k=1}^{n} g_k f_k = e. \tag{1.7}$$

In this language, the Wiener-Pitt phenomenon is exactly the 1-invisibility of the spectrum for the measure algebra $\mathcal{M}(G)$, if we stay on the dual group $X = \hat{G}$. Rigorously speaking, we mean the algebra of Fourier transforms $\mathcal{F}M(G) = \{\hat{\mu} : \mu \in \mathcal{M}(G)\}$ endowed with the norm $\|\hat{\mu}\| = \|\mu\|$ and embedded into $C(\hat{G})$. In what follows, we systematically identify these algebras.

The next definition specifies the previous one in the case of norm controlled invertibility instead of simple invertibility.

DEFINITION 1.2.3. Let A and X be as above, and let $0 < \delta \leqslant 1$. The spectrum of A is called $(\delta - n)$*-visible* (*from* X) if there exists a constant c_n such that any Bezout equation (1.7) with data $f = (f_1, \dots, f_n) \in A^n$ satisfying (1.6) and the normalizing condition

$$\|f\|^2 =: \sum_{k=1}^{n} \|f_k\|^2 \leqslant 1 \tag{1.8}$$

has a solution $g \in A^n$ with $\|g\| \leqslant c_n$. The spectrum is called *completely* δ*-visible* if it is *($\delta - n$)-visible for all $n \geqslant 1$ and the constants c_n can be chosen in such a way that* $\sup_{n \geqslant 1} c_n < \infty$.

Clearly, there exist the *best possible constants*, in the following sense. Setting

$$c_n(\delta) =: c_n(\delta, A) = c_n(\delta, A, X) = \sup_f \{\inf(\|g\| : \sum_{k=1}^{n} g_k f_k = e,\ g \in A^n)\}, \tag{1.9}$$

where the *supremum* is taken over all $f \in A^n$ satisfying (1.8) and (1.6), we get the smallest number for which $c_n = c_n(\delta) + \epsilon$ meets the requirements of Definition 1.2.3 for every $\epsilon > 0$. In particular,

$$c_1(\delta) = \sup\{\left\|f^{-1}\right\| : f \in A;\ \delta \leqslant |f(x)| \leqslant \|f\| \leqslant 1,\ x \in X\}, \tag{1.10}$$

and, in this case, we can take $c_1 = c_1(\delta)$ in Definition 1.2.3; here and in what follows we formally set $\|f^{-1}\| = \infty$ for noninvertible elements of A.

We define the n-th critical constant $\delta_n(A, X)$ by the relation

$$\delta_n(A, X) = \inf\{\delta : c_n(\delta, A, X) < \infty\}$$

1.2.4. **Main problems.** Our main objective is to estimate from above and from below, and (if possible) to compute the critical constants $\delta_n(A, X)$ and the majorants $c_n(\delta, A, X)$ for basic Banach algebras A and, thus, to study norm controlled visibility properties for these algebras.

For $n = 1$ we deal with more general objects. Namely, we are interested in describing functions boundedly acting on a given algebra. Supposing that a "visible part" of the spectrum is fixed, $X \subset \mathfrak{M}(A)$, we can say that a function (say, φ) defined on $\sigma \subset \mathbb{C}$ acts on an algebra A if for every $a \in A$ with $\hat{a}(X) \subset \sigma$ there exists $b \in A$ such that $\varphi \circ \hat{a} = \hat{b}$ on X. A "bounded action" is defined in the same way but adding an estimate of the form $\|b\| \leqslant k\|\varphi\|_*$, where $k = k(\sigma, A, X)$, and $\|\cdot\|_*$ is an appropriate norm majorazing $\|\varphi\|_\sigma = \sup_\sigma |\varphi|$. We mention, however, that such a definition is too broad to be useful: *the spectral inclusion $\hat{a}(X) \subset \sigma$ alone, without any norm restrictions, cannot imply the boundedness of compositions*. We formalize this statement as follows.

LEMMA 1.2.5. *Assume that $|\hat{a}(x)| \geqslant \delta (x \in X)$ always implies $\|a^{-1}\| \leqslant C$, for some positive δ and C. Then A is a uniform algebra whose norm is equivalent to the* $\sup$ *norm on X: $\|\hat{a}\|_X \leqslant \|a\| \leqslant (2C + \delta)\|\hat{a}\|_X$ for all $a \in A$.*

The lemma shows that certain norm requirements are necessary. The classical results on functions operating on Fourier transforms show many specific examples of this kind, see [HKKR], [Ru1] and further references therein.

In paper [N4], adding the normalizing condition $\|a\| \leqslant 1$ to the spectral inclusion $\hat{a}(X) \subset \sigma$ we look for a "minimal spectral hull" $h(\sigma) = h(\sigma, A, X)$ such that functions holomorphic on $h(\sigma)$ boundedly act on a given algebra. Our approach to this problem is explained in Section 2.

1.3. *Outline of the theory*

The main goal of the theory presented below is to estimate, from above and from below, and (if possible) to compute, the critical constants $\delta_n(A, X)$ and the majorants $c_n(\delta, A, X)$ for some commutative Banach algebras frequently used in harmonic analysis and for the corresponding (customary) visible parts X of their spectra. The basic algebras are the following ones:

(i) the measure algebra $\mathcal{M}(G)$ on an infinite LCA group G with $X = \hat{G}$, and in particular, the Wiener algebra $W = \mathcal{F}l^1(\mathbb{Z})$ of absolutely convergent Fourier series with $X = \mathbb{T}$ (Section 5);

(ii) the analytic Wiener algebra $W_+ = \mathcal{F}l^1(\mathbb{Z}_+)$ with $X = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ (Section 5);

(iii) the weighted Beurling-Sobolev algebras of positive spectral radius, both analytic $\mathcal{F}l^p(\mathbb{Z}_+, w)$ (with $X = \overline{\mathbb{D}}$), and "symmetric" $\mathcal{F}l^p(\mathbb{Z}, w)$ (with $X = \mathbb{T}$), for some class of regularly growing weights $w(n)$ tending to ∞ as $|n| \longrightarrow \infty$ (Section 7).

In Section 2, following [N4], we introduce two general methods: a method for upper estimates of the visibility constants $c_n(\delta, A, X)$, and a method for their lower estimates. The method for lower estimates for $c_n(\delta, A, X)$ described in Subsection 2.3 refers to elements of A with "almost independent" powers. In Section 5 we specify this method for the algebras $\mathcal{M}(G)$, $\mathcal{M}(S)$ using the Sreider measures, which are defined as measures with real Fourier transforms and the spectrum filling in a disc. In particular, a short proof is given, following [N4], to the fact that $\delta_1(W_+, \overline{\mathbb{D}}) = 1/2$ and $c_1(\delta) = (2\delta - 1)^{-1}$ for $1/2 < \delta \leqslant 1$ and $c_1(\delta) = \infty$ for $\delta \leqslant 1/2$. A different, longer but more elementary proof of this fact is contained in [ENZ]. Independently, another elementary proof of the equality $\delta_1(W_+, \overline{\mathbb{D}}) = 1/2$ was recently presented by H.S. Shapiro at the 6th St. Petersburg Summer Analysis Conference, see [Sh2].

In the same Section 5, we show that $c_1(\delta, \mathcal{F}M(G), \hat{G}) = \infty$ for $0 < \delta \leqslant 1/2$, and thus $\delta_1(\mathcal{F}M(G), \hat{G}) \geqslant 1/2$, for every infinite LCA group G. Moreover, $(2\delta - 1)^{-1} \leqslant c_1(\delta, \mathcal{F}M(G), \hat{G}) \leqslant c_n(\delta, \mathcal{F}M(G), \hat{G})$ for $1/2 < \delta \leqslant 1$, and $c_n(\delta, \mathcal{F}M(G), \hat{G}) \leqslant (2\delta^2 - 1)^{-1}$ for $1/\sqrt{2} < \delta \leqslant 1$. Again, these results are contained both in [N4] and [ENZ] but the proofs are different.

It should be noted that some of these results were mentioned even in Shapiro's paper [Sh1] (Remarks 2 and 3, and a footnote on page 235 of [Sh1]), where they were attributed to Y. Katznelson (for $c_1(\delta, W, \mathbb{T}) = \infty$ for $\delta < 1/2$), to Y. Katznelson and D. J. Newman (for $c_1(\delta, W, \mathbb{T}) < \infty$ for $\delta > 1/\sqrt{2}$), and to Bell (for $c_1(\delta, W_+, \overline{\mathbb{D}}) < \infty$ for $\delta > 1/2$), but at present we cannot specify references.

Section 6 contains some results on the efficient inversion on some finite groups and semigroups (again, following [N4]).

In Section 4, we deal with a more general framework, namely with the norm-controlled functional calculi and the so-called spectral efficiency hulls $h(\sigma, A, X)$ introduced (following [N4]) in the same section. The links of efficiency hulls with the norm-controlled calculi are established and a description of these hulls for the measure algebra $\mathcal{M}(S)$ on a semigroup S is given using the horodisc expansions.

The results and the methods employed for *weighted* convolution algebras are completely different. In Section 3, following [ENZ], a general method is developed that allows us to control the inverses in terms of $\delta = \inf_{\mathfrak{M}(A)} |\hat{x}|$ for rotation invariant topological Banach algebras A. More precisely, our goal in Section 3 is to give a method for proving the equality $\delta_1(A, \mathfrak{M}) = 0$. This method relies on two ideas. First, to estimate $\|x^{-1}\|_A$, we use the multipliers $\mathrm{mult}\,(\mathcal{D}A)$ of the space $\mathcal{D}A$ of derivatives of our algebra A, where $\mathcal{D} = z\frac{d}{dz}$. The second idea is to deduce estimates from the compactness of the embedding $A \subset \mathrm{mult}\,(\mathcal{D}A)$. In applications, the main point is precisely in the proof of this compactness and in estimating the rate of decay of relevant best polynomial approximations in the $\mathrm{mult}\,(\mathcal{D}A)$ norm. Following [ENZ], we realize this approach in Section 7 for the Beurling-Sobolev algebras $A = l^p(w)$ of positive spectral radius, $r(A) = \lim_n w(n)^{1/n} > 0$, both on $\mathbb{Z}$ and $\mathbb{Z}_+$.

Historical remarks. As is already mentioned, the prehistory of the ideas presented in this paper was started with the classical theorems of Wiener-Lévy and Wiener-Pitt-Sreider quoted above.

The second wave of results, sharpening the Gelfand and Riesz-Dunford functional calculi, was devoted to functions operating on Fourier transforms, and was mostly due to H. Helson, J.-P. Kahane, Y. Katznelson, and W. Rudin. The main problem considered and resolved was to describe functions φ defined on an interval $\sigma = [a, b] \subset \mathbb{R}$ and such that $\varphi(\mathcal{F}f) \in \mathcal{F}A$ for every $f \in A$ with $\mathcal{F}f(\hat{G}) \subset \sigma$, where $A = \mathcal{M}(G)$ or $A = L^1(G)$. See [HKKR], [Ru1], [GMG], [K1], [HR] for exhaustive presentations and further references. Nonanalytic functions operating on certain weighted algebras of Fourier transforms $\mathcal{F}l^1(\mathbb{Z}, w)$ occurred in the papers of J.-P. Kahane, [K2], and N. Leblanc [L]. However, no quantitative aspects similar to those of Sections 1–2 were explicitly presented.

The third wave of results related to norm-controlled calculi can be linked with constructive proofs of the Wiener-Lévy theorem on inverses. We mention the proofs by A. Calderon, presented in [Z], by P. Cohen [C], and by D. Newman [New]. The Calderon approach was developed by E. Dyn'kin [D].

The problem of norm-controlled inversion (for the Wiener algebra $A = l^1(\mathbb{Z})$) was first mentioned by J. Stafney in [St], where the existence of $a, b, K > 0$ was proved such that $\sup\{\|f^{-1}\|_A : \|f\| \leqslant K, \hat{f}(\mathbb{T}) \subset [a, b]\} = \infty$. This implies that $c_1(\delta, A, \mathbb{T}) = \infty$ for *some* $\delta > 0$. The proof, based on Y. Katznelson's results, does not permit to specify the value of δ. Independently, and in a more constructive way, the result was obtained by H. Shapiro [Sh1] (in response to a question by a physicist G. Ehrling), but also without any concrete value of δ. Several remarks were made in Shapiro's note attributing to various authors certain estimates for quantities we call the critical constants $\delta_1(l^1(\mathbb{Z}_+), \overline{\mathbb{D}})$ and $\delta_1(l^1(\mathbb{Z}), \mathbb{T})$; no precise references were given. In fact, the paper [N4] was inspired by Shapiro's construction. In the paper of J.-E. Björk [B], a problem related to uniform functional calculi was considered. In our language, it is equivalent to an estimate for $\delta_1^0(A, \mathfrak{M}(A))$, a microlocal version of $\delta_1(A, \mathfrak{M}(A))$ studied in Section 4 below. In [B], a criterion was given in terms of another quantity which can be regarded as a "uniform spectral radius" of the algebra. O. El-Fallah [E] recently gave an application of Bkörk's result to the algebras $l^p(w)$ with slowly growing w.

In the paper of S. Vinogradov and A. Petrov [VP], a description was given of Banach spaces A of functions on $\mathbb{T}$ satisfying the following property: $f \in A$ and $|f| \geqslant \delta$ on $\mathbb{T}$ imply $1/f \in A$.

We finish these remarks mentioning that the case of higher Bezout equations (the "corona problems") related to the constants $c_n(\delta, A, X)$ and $\delta_n(A, X)$ for $n > 1$ is considered in this paper very briefly. For their history see [Gar], [N1], as well as [To1] and [To2].

2. How and Why One Can(not) Control the Inverses

Let A be a commutative Banach algebra with unit e, and let X be a Hausdorff topological space such that A is continuously embedded into $C(X)$, so that $X \subset \mathfrak{M}(A)$. We also use other notation introduced in Section 1. The set of invertible elements of A is denoted by $\mathcal{G}(A)$. We start with simple observations on the critical constant δ_1.

2.1. *First observations*

We say that A is an algebra of *distance controlled resolvent growth* if there exists a monotone decreasing function $\varphi : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^* = (0, \infty)$ such that

$$\left\|(\lambda e - f)^{-1}\right\| \leqslant \varphi(\mathrm{dist}\,(\lambda, \sigma(f)), \quad \lambda \in \mathbb{C} \backslash \sigma(f) \tag{2.1}$$

for all $f \in A, \|f\| \leqslant 1$. It is easy to see that this estimate implies $\|(\lambda e - f)^{-1}\| \leqslant \frac{1}{\|f\|}\varphi(\frac{1}{\|f\|}\mathrm{dist}\,(\lambda, \sigma(f)))$ for all $f \in A$ and $\lambda \in \mathbb{C}\backslash\sigma(f)$. It is shown in [N4] that a commutative Banach algebra A obeys the distance controlled resolvent growth if and only if $\delta_1(A, \mathfrak{M}(A)) = 0$ (the critical constant introduced in Section 1). In fact, $c_1(\delta, A, \mathfrak{M}) \leqslant \varphi(\delta) \leqslant c_1(\delta(2 + \delta)^{-1}, A, \mathfrak{M})$. Similar equivalences hold for the constant $\delta_1(A, X)$ related to a subset $X \subset \mathfrak{M}(A)$, and for $(\delta - n)$-visibility and the "n-resolvent" $(\lambda e_n - f)^{-1}$, where $\lambda e_n = (\lambda_1 e, \ldots, \lambda_n e) \in A^n$, $\lambda_k \in \mathbb{C}$.

For many examples of function Banach algebras with various behaviour of constants δ_n and $c_n(\delta)$, including the classical ones (like $A = H^\infty(\Omega)$), we refer to [N4], [ENZ].

2.2. *Splitting X-symmetric algebras for upper estimates*

As before, we consider a commutative unital Banach algebra A continuously embedded into the space $C(X)$, where $X \subset \mathfrak{M}(A)$. Now, we need the following definition, see [N4].

DEFINITION 2.2.1. We say that an algebra A *splits at the unit* if there exists a subspace $A_0 \subset A$ such that $A = e \cdot \mathbb{C} + A_0$ (a direct sum) and for $f = \lambda e + f_0$, $f_0 \in A_0$ we have $\|f\| = |\lambda| + \|f_0\|$.

An algebra A is said to be X-*symmetric* if for every $f \in A$ there exists an element $g \in A$ such that $\|g\| \leqslant \|f\|$ and $g(x) = \overline{f(x)}$ for all $x \in X$.

Obviously, the splitting property is satisfied by the algebras A obtained by the standard adjoining of unity to a Banach algebra A_0 without unit. For instance, this is the case for the group algebra $A_0 = L^1(G)$ of a nondiscrete LCA group G. Also, it should be mentioned that the classical symmetry property of Banach algebras corresponds, in our language, to the $\mathfrak{M}(A)$-symmetry. For $X \neq \mathfrak{M}(A)$, X-symmetry may happen to be a considerably weaker property. For instance, the algebra $A = \mathcal{F}M(G)$ is obviously $\hat{G}-$symmetric: for $f = \hat{\mu} \in A$

take $g = \hat{\mu}_*$, where $\mu_*(\sigma) = \overline{\mu(-\sigma)}, \sigma \subset G$. But it is not $\mathfrak{M}$-symmetric for a nondiscrete LCA group G. This latter property is essentially equivalent to the Wiener-Pitt phenomenon.

For X-symmetric algebras, the ($\delta - n$)-visibility properties are related to each other as follows.

LEMMA 2.2.2. *[N4] For an X-symmetric algebra A, the ($\delta - 1$)-visibility of the spectrum implies the $(\sqrt{\delta} - n)$-visibility for all $n \geqslant 1$, and even the complete $\sqrt{\delta}-$visibility with* $\sup_{n\geqslant 1} c_n(\sqrt{\delta}, A, X) \leqslant c_1(\delta, A, X)$.

Another feature of X-symmetric algebras is that one can always control the inverses of elements whose lower bound $\delta = \inf_X |f|$ is sufficiently close to the norm $\|f\|$. To show this we use the following obvious observation.

LEMMA 2.2.3. *Let A be an algebra splitting at the unit, and let $f = \lambda e + f_0$, $1/2 < \delta \leqslant |\lambda| \leqslant \|f\| \leqslant 1$. Then f is invertible in A, and $\|f^{-1}\| \leqslant (2\delta - 1)^{-1}$.* ∎

2.2.4. **X-domination for the unit evaluation functional.** Let φ_e be the following functional evaluating the coefficient of the unit in the standard expansion of an element of a splitting algebra:

$$\varphi_e(\lambda e + f_0) = \lambda \tag{2.2}$$

for $\lambda e + f_0 \in A = \mathbb{C}{\cdot}e + A_0$. Note that φ_e is a norm 1 linear functional on A, not necessarily multiplicative. The following definition will be useful.

DEFINITION 2.2.5. Let $A = e{\cdot}\mathbb{C} + A_0$ be a direct sum decomposition of a Banach algebra A, and let $X \subset \mathfrak{M}(A)$. We say that the unit evaluation functional (2.2) is X-*dominated* if $|\varphi_e(f)| \leqslant \|f\|_X$ for every $f \in A$, where $\|f\|_X = \sup_X |f|$.

Standard Hahn-Banach arguments show the following lemma, where, as above, δ_x means the evaluation functional at a point $x \in X$: $\delta_x(f) = f(x)$ for $f \in A$.

LEMMA 2.2.6. *The following assertions are equivalent.*

(i) The functional φ_e of (2.2) is X-dominated.

(ii) $\varphi_e \in \operatorname{conv}(\delta_x : x \in X)$, where $\operatorname{conv}(\cdot)$ stands for the weak-$*$ closed convex hull of $(\cdot)$. ∎

THEOREM 2.2.7. *Let A be a commutative unital Banach algebra and $X \subset \mathfrak{M}(A)$ such that (i) A is X-symmetric; (ii) A splits at the unit; and (iii) φ_e is X-dominated.*

Then, the spectrum of A is $(\delta - n)$-visible for all $n \geqslant 1$ and all δ satisfying $1/\sqrt{2} < \delta \leqslant 1$, and even completely δ-visible with $c_n(\delta, A, X) \leqslant (2\delta^2 - 1)^{-1}$.

Proof. Let $f = (f_1, \ldots, f_n) \in A^n$ be such that $\delta \leqslant |f(x)| \leqslant \|f\| \leqslant 1$ for all $x \in X$, and let g_k be elements of A corresponding to the f_k as in the definition of X-symmetric algebras. Then $h \in A$, where $h = \sum_{k=1}^{n} f_k g_k$, and $1/2 < \delta^2 \leqslant h(x) \leqslant \|h\| \leqslant 1$ for all $x \in X$. Using condition *(iii)*, Lemma 2.2.6, and an obvious fact that the interval $[\delta^2, 1]$ is a convex set, we obtain $\delta^2 \leqslant \varphi_e(h) \leqslant 1$. Condition *(ii)* and Lemma 2.2.3 imply that h is invertible and $\|h^{-1}\| \leqslant (2\delta^2-1)^{-1}$. Hence, $g = (g_1h^{-1}, \dots, g_nh^{-1}) \in A^n$, $\sum_{k=1}^n f_k(g_kh^{-1}) = e$, and

$$\|g\| = (\sum_{k=1}^n \|g_kh^{-1}\|^2)^{1/2} \leqslant \|h^{-1}\| \cdot \|f\| \leqslant (2\delta^2-1)^{-1},$$

as desired. ∎

2.3. *A method for lower estimates*

The method to get a lower estimate for $c_1(\delta, A, X)$ stated in theorem 2.3.1 below is inspired by Shapiro's example [Sh1] mentioned above. Essentially, it reduces to the existence of elements $a \in A$ whose normalized powers $a^k/\|a\|^k$, $0 \leqslant k \leqslant p$, for a given p are ϵ-equivalent to the standard basis of an l^1-space, whereas asymptotically they tend to zero faster than a given exponential. We use this method in Section 5; see also [ENZ].

THEOREM 2.3.1. *[N4] Let A be a unital commutative Banach algebra, $X \subset \mathfrak{M}(A)$, and given $\epsilon > 0$ let A^ϵ be the set of all elements $a \in A$ such that $\|a\|_{C(X)} < \epsilon$ and $\|a\| = 1$. Suppose that*

$$\sup_{a\in A^\epsilon} \left\| \sum_{k=0}^p b_k a^k \right\| \geqslant \sum_{k=0}^p b_k \tag{2.3}$$

for all $p \geqslant 0$, $\epsilon > 0$, and $b_k \geqslant 0$. Then $c_1(\delta, A, X) \geqslant (2\delta-1)^{-1}$ for all δ, $1/2 < \delta < 1$. In particular, $\delta_1(A, X) \geqslant 1/2$.

The proof consists of setting $f_t = (1+t)^{-1}(e - ta)$ for $t > 0$ such that $(1+t)^{-1} > \delta$ and $1/2 < \delta < 1$, then estimating $\|f_t^{-1}\|$ from below and maximizing in t. See [N4], theorem 1.5.1, for details.

3. Estimates of Inverses for Rotation Invariant Algebras

In this Section we describe, following [ENZ], a general method permitting to control the inverses in terms of $\delta = \inf_{\mathfrak{M}(A)} |\hat{x}|$ for rotation invariant topological Banach algebras A on the circle $\mathbb{T}$. In other words, our goal is to give a method for proving the equality $\delta_1(A, \mathfrak{M}) = 0$. As mentioned above, the latter property is equivalent to a distance controlled estimate for resolvents, $\|(\lambda e - x)^{-1}\| \leqslant \varphi(\mathrm{dist}\,(\lambda, \sigma(x))$.

In short, the main idea how to estimate $\|x^{-1}\|$, $x \in A$, is to "reduce the smoothness" of x^{-1} by applying a first order differential operator $\mathcal{D}$, and then to use the formula $\mathcal{D}x^{-1} = -x^{-2}\mathcal{D}x$. To estimate the norm of the product $x^{-2}\mathcal{D}x$ we use the range norm $\|\cdot\|_{\mathcal{D}A}$ on $\mathcal{D}A$ and the multiplier norm related to $\mathcal{D}A$; namely $\|\mathcal{D}x^{-1}\|_{\mathcal{D}A} \leqslant \|\mathcal{D}x\|_{\mathcal{D}A}\|x^{-2}\|_{\mathrm{mult}\,(\mathcal{D}A)}$.

The second idea is to rely on the compactness of the embedding $A \subset \mathrm{mult}\,(\mathcal{D}A)$ to ensure a uniform estimate for $\|x^{-2}\|_{\mathrm{mult}\,(\mathcal{D}A)}$. We obtain an estimate for $c_1(\delta, A)$ measuring the size of a compact subset of $\operatorname{mult}(\mathcal{D}A)$ in terms of the decreasing rate of the best polynomial approximations, see Subsection 3.2 below.

In Section 7 we apply this method to Beurling-Sobolev algebras

$$A = l^p(\mathbb{Z}, w), l^p(\mathbb{Z}_+, w).$$

3.1. *How to use multipliers*

Having in mind applications to the case $A = l^p(w)$ (Section 7), from now on we distinguish between the Banach algebras and the algebras that become a Banach algebra after equivalent norming. More precisely, *a unital Banach algebra* is a Banach space A endowed with a multiplication such that $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in A$, and $\|e\| = 1$, e stands for the unit of A. A *unital topological Banach algebra* A is a Banach space A endowed with a continuous multiplication so that $\|xy\| \leqslant C\|x\| \cdot \|y\|$ for all $x, y \in A$ and for some constant C. Clearly, every commutative topological Banach algebra is a Banach algebra with respect to the operator norm $\| \cdot \|^*$,

$$\|x\|^* = \sup\{\|xy\| : y \in A,\ \|y\| \leqslant 1\}.$$

It is clear that some properties of A, such as, e.g., the property $\delta_1(A, X) = 0$, are renorming stable. Some others may be greatly affected by such a renorming. For instance, so is the sharp value of the critical constant $\delta_1(A, X)$ if it is positive, or, in the case where $\delta_1(A, X) = 0$, so is the growth rate of the constants $c_1(\delta, A, X)$ as $\delta \longrightarrow 0$.

Now, let A be a unital topological Banach algebra of sequences $x = (x_n)$ on $\mathbb{Z}$ or $\mathbb{Z}_+$, which means that $x \longmapsto x_n$ is a continuous functional for every n, with the convolution $*$ as an algebra operation, and such that

(i) the set $\mathcal{S}_0$ of finitely supported sequences (on $\mathbb{Z}$ or $\mathbb{Z}_+$, respectively) is a dense subset of A;

(ii) A is a rotation invariant (homogeneous) space of sequences, that is, if $x \in A$ then $x_t = (x_n\overline{t}^n)_n \in A$ and $\|x_t\| = \|x\|$ for every $t \in \mathbb{T}$.

Conditions *(i)* and *(ii)* guarantee that the rotation $t \longmapsto x_t$ is a norm continuous mapping from $\mathbb{T}$ to A, for every $x \in A$. Consequently, the Césaro (Fejér) or Abel-Poisson averages of the series $x \sim \sum_k x_k e_k$ converge to x for every $x \in A$; here $e_k = (\delta_{kn})_n$ is the standard $0-1$ algebraic basis of $\mathcal{S}_0$.

A complex homomorphism φ of A is uniquely determined by its value $\lambda = \varphi(e_1)$ on the generator e_1 of the algebra A, and the Gelfand (Fourier) transformation is given by the formula,

$$x \longmapsto \hat{x}(\lambda) = \mathcal{F}x(\lambda) = \sum_k x_k \lambda^k,$$

at least for $x \in \mathcal{S}_0$. Hence, $\varphi \longmapsto \lambda = \varphi(e_1)$ is a bijection of $\mathfrak{M}(A)$ on a compact subset of $\mathbb{C}$. We identify this subset with $\mathfrak{M}(A)$. In can be easily seen that $\mathfrak{M}(A) = A(r_-, r_+) = \{z \in \mathbb{C} : r_- \leqslant |z| \leqslant r_+\}$, where $r_+ = \lim_n \|e_n\|^{1/n} < \infty$ and $r_- = \lim_n \|e_{-n}\|^{-1/n} > 0$. We refer to $r_\pm = r_\pm(A)$ as to the lower and the upper spectral radius of A. The normalized case, where $r_+ = 1$, or $r_+ = r_- = 1$ in the case of $\mathbb{Z}$, will be the main one for us.

3.1.1. **A Green type norm.** The *operator* $\mathcal{D}$ is defined by $\mathcal{D}x = (nx_n)_n$, $x \in A$. It maps A to $\mathcal{S}$, the space of all sequences on $\mathbb{Z}$ (respectively, on $\mathbb{Z}_+$). On the Gelfand transforms $\hat{x} = \sum_k x_k z^k$, the operator $\mathcal{D}$ acts as $\hat{\mathcal{D}}\hat{x} = (\mathcal{D}x)\hat{}$. This is a formal first order differential operator, $\hat{\mathcal{D}} = z\frac{d}{dz}$. In particular, it obeys the Leibnitz rule for products $\hat{\mathcal{D}}(fg) = f\hat{\mathcal{D}}g + g\hat{\mathcal{D}}f$, at least for "trigonometric polynomials" $f, g \in \hat{\mathcal{S}}_0$. Hence, $\mathcal{D}(x * y) = x * \mathcal{D}y + y * \mathcal{D}x$ for every $x, y \in \mathcal{S}_0$. In fact, the same formula works for $x \in \mathcal{S}_0$, $y \in A$, and in particular, $\mathcal{D}x^{-1} = -x^{-2} * \mathcal{D}x$ for all $x \in \mathcal{G}(A) \cap \mathcal{S}_0$.

The range $\mathcal{D}A$ is a Banach space with respect to the range norm $\|\mathcal{D}x\|_{\mathcal{D}A} = \|x\|_A$, where $x \in A$, $x_0 = 0$. For convenience reasons, we add the unit $e = e_0$ to $\mathcal{D}A$ and will consider the sum $A' = \mathcal{D}A + \mathbb{C}\cdot e$ as the *range space of* $\mathcal{D}$ endowed with the following range norm $\|\lambda e + \mathcal{D}x\|_{A'} = \|\lambda e + x\|_A$, where $x \in A$, $x_0 = 0$ and $\lambda \in \mathbb{C}$. Clearly, the space A' contains $\mathcal{S}_0$ as a dense subset, and is rotation invariant.

It is shown, see [ENZ], that $A \subset A'$ and every element $x \in \mathcal{S}_0$ defines a continuous convolution operator $y \longmapsto x * y$ on A'. Hence, one can introduce the *convolution multiplier norm* on finitely supported elements $x \in \mathcal{S}_0$ as the operator norm,

$$\|x\|_{\text{mult}\,(A')} = \sup\{\|x * y\|_{A'} : y \in \mathcal{S}_0,\ \|y\|_{A'} \leqslant 1\}.$$

By definition, the *space* $\text{mult}\,(A')$ *of (little) convolution multipliers* of A' is the completion of $\mathcal{S}_0$ with respect to this norm. Clearly, $\text{mult}\,(A')$ is a unital rotation invariant Banach algebra.

LEMMA 3.1.2. *Let A be a topological Banach algebra satisfying the above conditions (i)–(ii), and let $x \in \mathcal{G}(A) \cap \mathcal{G}(\text{mult}\,(A'))$ such that $\delta = \min_{\lambda \in \mathfrak{M}(A)} |\hat{x}(\lambda)| > 0$. Then*

$$\|x^{-1}\|_A \leqslant \|e\|_A \delta^{-1} + 2\|x\|_A \cdot \|x^{-2}\|_{\text{mult}\,(A')}.$$

3.2. *How to use compactness*

3.2.1. **A multiplier estimate.** Lemma 3.1.2 makes evident the following sufficient condition for *$(\delta - 1)$-visibility of the spectrum*: given a topological Banach algebra A of sequences on $\mathbb{Z}$, or $\mathbb{Z}_+$, satisfying conditions *(i)–(ii)* of Subsection 3.1, and *compactly* embedded into $\text{mult}\,(A')$: $A \subset_c \text{mult}\,(A')$, then $\delta_1(A, \mathfrak{M}(A)) = 0$, and, moreover,

$$c_1(\delta, A, \mathfrak{M}) \leqslant \delta^{-1}\|e\|_A + 2C(K_\delta)$$

for all $\delta > 0$, where $K_\delta = A_\delta^2$, $A_\delta = \{x \in A : \|x\|_A \leqslant 1, |\hat{x}(\lambda)| \geqslant \delta \text{ for all } \lambda \in \mathfrak{M}(A)\}$ and $C(K_\delta)$ is defined by the formula

$$C(K_\delta) = \sup\{\left\|x^{-1}\right\|_{\text{mult}\,(A')} : x \in K_\delta\} < \infty. \tag{3.1}$$

It remains to estimate the constant $C(K_\delta)$ of formula (3.1).

3.2.2. **Approximate characteristics of compact sets.** We use the standard classification of compact sets in terms of the best polynomial approximations. Let B be a Banach space and let $L_n \subset B$ be subspaces of B such that $L_n \subset L_{n+1}$, $dimL_n < \infty$ for $n \geqslant 1$, and $\operatorname{clos}(\bigcup_n L_n) = B$. Further, let $K \subset B$ and

$$\epsilon_n(K) = \epsilon_n(K, B) = \sup\{\operatorname{dist}(x, L_n) : x \in K\} \tag{3.2}$$

be the best approximations of K by elements of L_n. It is well known that *a bounded subset $K \subset B$ is relatively compact if and only if* $\lim_n \epsilon_n(K) = 0$. We apply this criterion to the space $B = \operatorname{mult}(A')$ and to subspaces $L_n = \mathcal{P}_n$ of all trigonometric polynomials of degree less than or equal to n. Then $\epsilon_n(K)$ of (3.2) are the best polynomial approximations of elements of K. The axioms *(i)* and *(ii)* of Subsection 3.1 and the definition of $\operatorname{mult}(A')$ imply that $\lim_n \|\Phi_n x - x\|_B = 0$ for all $x \in B$, where $\Phi_n x$ stands for the Fejer mean of a sequence $x \in B$. Therefore, we can use the above compactness criterion for $B = \operatorname{mult}(A')$.

Now, our aim is to specify constants $C(K_\delta)$ from formula (3.1) in terms of $\epsilon_n(K_\delta)$, $n \geqslant 1$.

3.2.3. **Calderon-Cohen-Dyn'kin "constructive inversions".** As mentioned in the beginning of Section 3, the use of compactness for estimates of inverses was started with various "constructive proofs" of the classical Wiener-Lévy inversion theorem for absolutely convergent Fourier series. Here "constructive" means "without using the Gelfand theory of maximal ideals". The basic A. Calderon proof, [Z] Chapter VI, theorem (5.2), exploits the Cauchy formula, and, thus, is applicable not only to inverses x^{-1}, but also to compositions $\varphi \circ x$. P. Cohen [C] used the same techniques for inverses and for higher Bezout equations. In [C], a possibility to estimate the norms $\|x^{-1}\|_W$ in terms of $\delta = \inf_{\mathbb{T}} |\mathcal{F}x|$ and certain characteristics similar to the quantities ϵ_n of (3.2) was mentioned explicitly; here and below the symbol $\mathcal{F}x$ is used for the Gelfand (discrete Fourier) transform, keeping the notation $\hat{x}$ for the classical Fourier coefficients. In [D], E. Dyn'kin applied Calderon's method to the Beurling algebras $A = l^1(\mathbb{Z}, w)$ to get estimates for the inverses $\|x^{-1}\|_A$ in terms of the same characteristics $\epsilon_n(\{x\})$. D. Newman [New] gave a completely elementary proof of the Wiener-Lévy theorem also based on polynomial approximations.

All authors mentioned above obtained some estimates for the inverses $\|x^{-1}\|_A$ assuming, or implicitly assuming, that x runs over some compact subset $K \subset A$. The approach of [ENZ] is different: we *prove* the compactness of the set $A_\delta = \{x \in A : \|x\|_A \leqslant 1, |\mathcal{F}x| \geqslant \delta\}$ in the algebra $\operatorname{mult}(A')$ and use this compactness for obtaining a uniform estimate of $\|x^{-1}\|_A$ for $x \in A_\delta$. The next theorem is proved in [ENZ] using the Dyn'kin method from [D]. The latter paper contains a similar result for the special case of the algebra $B = l^1(\mathbb{Z}, w)$ with $w(n) = w(-n)$ and $r_- = r_+ = \lim_n w(n)^{1/n} = 1$.

THEOREM 3.2.4. *Let B be a convolution Banach algebra on $\mathbb{Z}_+$, or on $\mathbb{Z}$, satisfying conditions (i) and (ii) of Subsection 3.1, with the spectral radius r_+(respectively spectral radii $r_- \leqslant r_+$). Let K be a relatively compact subset of $B_\delta = \{x \in B : \|x\|_B \leqslant 1$, and $\delta \leqslant |\mathcal{F}x(\zeta)|$ for $\zeta \in \mathfrak{M}(B)\}$ and let $\lambda = \lambda_K$ be the distribution function of the sequence*

$(\epsilon_n(K))_{n\geqslant 0}$:

$$\lambda_K(t) = \text{card}\,\{n : n \geqslant 1,\ \epsilon_{n-1}(K) > t\}, \quad t > 0.$$

Then

$$\|x^{-1}\|_B \leqslant M(\delta,\lambda) = \frac{16}{\delta}\sum_{j\geqslant 0}(r_+^{-j}\|e_j\|_B + r_-^{j}\|e_{-j}\|_B)e^{-\delta j/17\lambda(\delta/4)}$$

for every $x \in K$.

For the case of an algebra on $\mathbb{Z}$ satisfying $r_- = r_+ = 1$, the proof starts by using the Calderon approach: we choose a polynomial $\mathcal{F}y \in \mathcal{P}_n$, $n = \lambda(\delta/4)$ with $\|x-y\|_B \leqslant \delta/4$ and write the Cauchy formula

$$x^{-1} = (2\pi i)^{-1}\int_{|z|=\delta/2}(y+ze)^{-1}(ze+(y-x))^{-1}\,dz,$$

where $(y+ze) \in \mathcal{G}(B)$ since $|\mathcal{F}y+z| \geqslant \delta/4$ on the space $\mathfrak{M}(B)$. To estimate $\|(y+ze)^{-1}\|_B$, we use Dyn'kin's method involving the S. Bernstein inequality, see [ENZ] for details.

Now, having in mind applications to the Beurling-Sobolev algebras in Section 7, we combine 3.2.1 and theorem 3.2.4.

THEOREM 3.2.5. *Let A be a convolution topological Banach algebra on $\mathbb{Z}$ or on $\mathbb{Z}_+$, satisfying conditions (i)–(ii) of Subsection 3.1 and compactly embedded into $B = \text{mult}\,(A')$, that is $A \subset_c B = \text{mult}\,(A')$. Let constants C and $\mathcal{E}$ be defined by the inequalities $\|x*y\|_A \leqslant C\|x\|_A\|y\|_A$ and $\|x\|_B \leqslant \mathcal{E}\|x\|_A$ for all $x, y \in A$, and let $\epsilon_n(A_0, B)$ be the best polynomial approximations of the unit ball $A_0 \subset A$, and $\lambda_0 = \lambda_{A_0}$ be their distribution function.*

Then $\delta_1(A, \mathfrak{M}(A)) = 0$, and $c_1(\delta, A, \mathfrak{M}(A)) \leqslant \frac{\|e\|_A}{\delta} + M(\delta)$ for all $\delta > 0$, where

$$M(\delta) = \frac{2^5\mathcal{E}^2}{\delta^2}\sum_{j\geqslant 0}(r_+^{-j}\|e_j\|_B + r_-^{j}\|e_{-j}\|_B)e^{-\delta^2 j/17\mathcal{E}^2\lambda_0(\delta^2/4C)}.$$

4. Spectral Hulls and Norm-Controlled Functional Calculi

In the three preceding sections, we considered the problem of uniform upper bounds for inverses when staying on a given subset $X \subset \mathfrak{M}$ of the maximal ideal space $\mathfrak{M}$. Following [N4], now we treat a more general form of the same problem defining and studying the so-called X-spectral efficiency hull $h(\sigma, X)$ of a given set $\sigma \subset \mathbb{C}$. In this language, the uniform boundedness of inverses is equivalent to the property $0 \notin h(\sigma_\delta, X)$, where σ_δ stands for the annulus $\{z \in \mathbb{C} : \delta \leqslant |z| \leqslant 1\}$. Yet another reason to study the hulls $h(\sigma, X)$ is that $h(\sigma, X)$ is the minimal set satisfying the "uniform calculus property". The latter means that for every open set $\Omega \subset \mathbb{C}$ containing $h(\sigma, X)$ there exists a constant $k = k(\Omega, X)$ such that $\|f(a)\| \leqslant k\|f\|_\Omega$ for every $f \in Hol(\Omega)$ and for every $a \in A$ with $\sigma(a) \subset \sigma$ and $\|a\| \leqslant 1$. Similar uniformly bounded calculi were implicitly involved in classical studies of functions operating on Fourier algebras, see [HKKR], [Ru1].

The main results of this section are theorems 4.1.5 and 4.2.2. Examples of spectral hulls are gathered in subsection 4.3.

4.1. *Spectral hulls and resolvent majorants*

Let A be a unital Banach algebra, and let $X \subset \mathfrak{M}(A)$.

DEFINITION 4.1.1. Let $\sigma \subset \overline{\mathbb{D}}$. We set $A(\sigma; X) = \{a \in A : \|a\| \leqslant 1,\ \hat{a}(X) \subset \sigma\}$, $C(\lambda, \sigma; X) = C(\lambda, \sigma; A, X) = \sup\{\|(\lambda e - a)^{-1}\| : a \in A(\sigma; X)\}$ for $\lambda \in \mathbb{C}$, where we take $\|(\lambda e - a)^{-1}\| = \infty$ for $\lambda \in \sigma(a)$. Let also $h(\sigma; X) = h(\sigma; A, X) = \{\lambda \in \mathbb{C} : C(\lambda, \sigma; X) = \infty\}$.

The set $h(\sigma; X)$ is called the *X-spectral hull of σ*; the *full spectral hull of σ* is $h(\sigma) = h(\sigma; A, \mathfrak{M}(A))$. The complement $\rho(\sigma; X) = \mathbb{C}\backslash h(\sigma; X)$ is called the *norm-controlled (or efficient) resolvent complement of σ*. For a positive constant $k > 0$, we also consider the sets $h(\sigma, k; X) = \{\lambda \in \mathbb{C} : C(\lambda, \sigma) > k\}$ and $\rho(\sigma, k; X) = \mathbb{C}\backslash h(\sigma, k; X)$. For $X = \mathfrak{M}(A)$ we simplify the notation in a natural way: $A(\sigma) = A(\sigma; \mathfrak{M}(A))$, $h(\sigma, k) = h(\sigma, k; \mathfrak{M}(A))$, $\rho(\sigma, k) = \rho(\sigma, k; \mathfrak{M}(A))$.

4.1.2. **First properties.** It is clear that the definition of the $(\delta - 1)$-*visibility* (Section 1) is a special case of those of the efficient resolvent complement. Indeed, let σ_δ be an annulus, $\sigma_\delta = \{z \in \mathbb{C} : \delta \leqslant |z| \leqslant 1\}$. Then the spectrum of A is ($\delta - 1$)-visible if and only if $0 \in \rho(\sigma_\delta; X)$. Moreover, $c_1(\delta, A, X) = C(0, \sigma_\delta; A, X)$, $0 < \delta \leqslant 1$.

Note that n-variables counterparts of $A(\sigma; X)$, $C(\lambda, \sigma; X)$, etc., could be considered as well, but we restrict ourselves to the case $n = 1$.

The following property of $C(\lambda, \sigma; A, X)$ and $h(\sigma, k; A, X)$ is a more or less straightforward consequence of the definitions, see [N4] for the proofs: $h(\sigma; X) = \sigma$ for every closed $\sigma \subset \overline{\mathbb{D}}$ if and only if $\delta_1(A, X) = 0$, where $\delta_1(A, X)$ is the critical constant for the pair (A, X).

4.1.3. **Microlocalization.** It is clear that lower estimates for spectral hulls and resolvent majorants must depend on the geometry of the subset $\sigma \subset \overline{\mathbb{D}}$ under consideration. For instance, $A(\sigma; X) = \{\mathrm{const}\}$ for every subset $X \subset \mathfrak{M}(A)$ equipped with an analytic structure and for every σ with $\mathrm{int}\,(\sigma) = \emptyset$, see 4.3.1 for examples. Having in mind this last constraint, we restrict ourselves to the case, where $\sigma = \mathrm{clos}\,(\mathrm{int}\,(\sigma))$.

In this case, the behaviour of $C(\lambda, \sigma; X)$ depends on the following microlocal version of the critical constants and the inversion majorants.

Let A be a unital Banach algebra, $X \subset \mathfrak{M}(A)$, and let $0 < \delta \leqslant 1$. A *microlocal upper bound* (*majorant*) for inverses is defined by

$$c_1^0(\delta, A, X) = \inf_{\epsilon > 0}(\sup\{\|f^{-1}\| : \|f\| \leqslant 1,\ \hat{f}(X) \subset \sigma_\delta,\ \mathrm{diam}\, \hat{f}(X) < \epsilon\}),$$

and the *microlocal critical constant* is defined by $\delta_1^0(A, X) = \inf\{\delta : c_1^0(\delta, A, X) < \infty\}$; here, as before, $\sigma_\delta = \{z \in \mathbb{C} : \delta \leqslant |z| \leqslant 1\}$.

Properties of these microlocal majorants are similar to those of the global ones, that is, to the properties of $\delta_1(A, X)$ and $c_1(\delta, A, X)$ defined in Sections 1 and 2; see [N4] for details.

4.1.4. **Horodisc expansions.** Denote $D(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$, $\overline{D}(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| \leqslant r\}$. In hyperbolic geometry of the unit disc $\mathbb{D}$, the discs $\overline{D}(z, 1-|z|)$, $z \in \overline{\mathbb{D}}$ are called *horodiscs* . Given a closed set $\sigma \subset \overline{\mathbb{D}}$, we call the set

$$\mathrm{hor}\,(\sigma) = \bigcup_{z\in\sigma} (\overline{D}(z, 1-|z|))$$

the *horodisc expansion* of σ. In order to apply the microlocal version of the critical constants we need one more notation, namely,

$$\mathrm{hor}\,(\sigma,\delta) = \bigcup_{z\in\sigma} (\overline{D}(z, \frac{\delta}{1-\delta}(1-|z|))).$$

THEOREM 4.1.5. *Let A be a unital splitting Banach algebra, X be a subset of $\mathfrak{M}(A)$ dominating φ_e, and let $\sigma = \mathrm{clos}\,(\mathrm{int}\,(\sigma)) \subset \overline{\mathbb{D}}$. Then*

(i) $\mathrm{hor}\,(\sigma, \delta_1^0) \subset h(\sigma; A, X) \subset \mathrm{hor}\,(\sigma, 1/2) = \mathrm{hor}\,(\sigma)$;

(ii) if $\delta_1^0(A, X) = 1/2$, then $h(\sigma; A, X) = \mathrm{hor}\,(\sigma)$;

(iii) if $\delta_1^0(A, X) = 1/2$ and $c_1^0(\delta, A, X) = (2\delta - 1)^{-1}$ (see Section 5 for examples), then $h(\sigma; A, X) = \mathrm{hor}\,(\sigma)$ and $C(\lambda, \sigma; X) = 1/\mathrm{dist}\,(\lambda, \mathrm{hor}\,(\sigma))$.

Observe that under the splitting condition we always have $\delta_1^0(A, X) \leqslant \delta_1(A, X) \leqslant 1/2$ and $\mathrm{hor}\,(\sigma, \delta_1^0) \subset \mathrm{hor}\,(\sigma, 1/2) = \mathrm{hor}\,(\sigma)$. The equality $\mathrm{hor}\,(\sigma, \delta_1^0) = \mathrm{hor}\,(\sigma)$ holds for all $\sigma \subset \overline{\mathbb{D}}$ if and only if $\delta_1^0(A, X) = 1/2$. For examples of computations and for pictures of the horodisc expansions of various sets see [N4].

4.2. *Spectral hulls and norm-controlled calculi*

The Gelfand theory guarantees the existence of a holomorphic calculus on the spectrum of every element of a Banach algebra A. Namely, if $a \in A$, the function of a

$$f(a) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)(\lambda e - a)^{-1}\,d\lambda$$

is well defined for every $f \in Hol(\overline{\Omega})$ and every open neighbourhood of the spectrum $\Omega \supset \sigma(a)$ (the Riesz-Dunford calculus). As is shown in Sections 1, 2, and 5, this does not guarantee any estimate of the norm $\|f(a)\|$, even for the simplest functions like $f(z) = 1/z$, and even if we add the normalizing condition $\|a\| \leqslant 1$ to the spectral inclusion $\sigma(a) \subset \Omega$.

The true question is the following: what are the relations between the spectrum $\sigma(a)$ (or a visible part of the spectrum $\hat{a}(X)$) and a domain Ω that guarantee the uniform continuity of the Ω-calculus? In other words, does there exist a compact set $K \subset \Omega$ and a constant $c > 0$ such that $\|f(a)\| \leqslant c \cdot \sup_K |f|$ for every $f \in Hol(\Omega)$ and every $a \in A(\sigma; X) = \{a \in A : \hat{a}(X) \subset \sigma, \|a\| \leqslant 1\}$? We formalize this setting in the following definition, see [N4] for more details.

DEFINITION 4.2.1. Let A and X be as above, and let $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$. We say that an open set $\Omega \subset \mathbb{C}$ *X-dominates the set* σ, or *is a norm-controlled calculus domain for* σ, if $\sigma \subset \Omega$ and the calculi $f \longmapsto f(a)$, $f \in Hol(\Omega)$ are well defined and uniformly continuous for all $a \in A(\sigma; X) = \{a \in A : \hat{a}(X) \subset \sigma, \|a\| \leqslant 1\}$.

In fact, this property is equivalent to the existence of a compact set $K \subset \Omega$ and a constant $c(K) = c(K, X) > 0$ such that

$$\|f(a)\| \leqslant c(K)\|f\|_K$$

for every function $f \in Hol(\Omega)$ and every $a \in A(\sigma; X)$. Here $\|f\|_K = \max\{|f(z)| : z \in K\}$.

Obviously, the definition of $(\delta - 1)$-visibility, as well as the definition of the distance controlled resolvent growth, see Subsection 1.1, are special cases of the latter concept. The following theorem shows that X-dominating domains for a given set σ can be described in terms of the spectral hulls $h(\sigma; X)$.

THEOREM 4.2.2. *Let A be a unital Banach algebra, $X \subset \mathfrak{M}(A)$, and let $\sigma \subset \overline{\mathbb{D}}$ and $\Omega \subset \mathbb{C}$ be a closed and an open set, respectively. The following assertions are equivalent.*

(i) Ω is an X-dominating domain for σ.

(ii) $\Omega \supset h(\sigma; X)$.

(iii) There exists $k > 0$ such that $\Omega \supset h(\sigma, k; X)$.

4.3. *Examples of spectral hulls*

Here we describe examples of three different types. As above, we refer to [N4] for details.

The first type pertains to algebras A whose hull operation is trivial in the sense that $h(\sigma; \mathfrak{M}(A)) = \sigma$ for every $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$; see 4.3.5 below.

For algebras A of the second type, the same operation $\sigma \longmapsto h(\sigma; \mathfrak{M}(A))$ is also trivial, but in a different way, namely, $h(\sigma; \mathfrak{M}(A)) = \overline{\mathbb{D}}$ for every nonempty $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$, even for singletons; see 4.3.4 below.

For the middle type algebras the full spectral hull $h(\sigma; \mathfrak{M}(A))$ essentially depends on σ but is different from it. For instance, this is the case for the measure algebras $A = \mathcal{M}(S)$ on semigroups considered in Section 5 below. In this case we can compute completely the full spectral hulls $h(\sigma; \hat{S}_b)$ and the resolvent majorants $C(\cdot, \sigma; \hat{S}_b)$, see theorem 4.3.2 below.

4.3.1. **Spectral hulls for measure algebras on semigroups.** Anticipating the systematic study of measure algebras (see Section 5 below), here we apply the above theory to compute relevant spectral hulls. Let $\mathcal{M}(S)$ be the convolution algebra of measures on a sub-semigroup of a locally compact abelian group G, and $\hat{S}_b$ be the set of all bounded semi-characters on S which we consider as the visible part of the spectrum of $\mathcal{M}(S)$ (see Section 5 for details). The corresponding Gelfand (Fourier-Laplace) transformation is denoted by $\mathcal{F}$. For example, the analytic Wiener algebra $A = \mathcal{F}M(\mathbb{Z}_+) = \mathcal{F}l_1(\mathbb{Z}_+) = W_+$ corresponds to $S = \mathbb{Z}_+$ and $\hat{S}_b = \mathfrak{M}(A) = \overline{\mathbb{D}}$, with $f \longmapsto f(z)$, $z \in \overline{\mathbb{D}}$, as the Gelfand transformation. Since a nonconstant holomorphic function is an open mapping, the sets $A(\sigma) = A(\sigma, \overline{\mathbb{D}}) = \{f \in W_+ : \|f\| \leqslant 1,\ f(\overline{\mathbb{D}}) \subset \sigma\}$ and $A(\sigma')$, where $\sigma' = \mathrm{clos}\,(\mathrm{int}\,(\sigma))$, differ from each other by constant functions taking values in $\sigma \backslash \sigma'$. Hence, we can restrict ourselves to the case, where $\sigma = \sigma'$. Supposing $\sigma = \sigma'$, we can easily deduce from theorem 4.1.5 a description of $h(\sigma; \hat{S}_b)$ for $\mathcal{M}(S)$, as well as for all subalgebras of $\mathcal{M}(S)$ considered in Section 5.

THEOREM 4.3.2. *[N4] Let $A = \mathcal{F}M(S)$, or let A be any algebra $A \subset \mathcal{M}(S)$ satisfying the conditions of theorem 4.3.4 below. Let σ be a closed subset of $\mathbb{D}$ such that $\sigma = \mathrm{clos}\,(\mathrm{int}\,(\sigma))$. Then*

(i) $h(\sigma; \hat{S}_b) = \mathrm{hor}\,(\sigma)$;

(ii) $C(\lambda, \sigma; \hat{S}_b) = 1/\mathrm{dist}\,(\lambda, \mathrm{hor}\,(\sigma))$ for $\lambda \in \mathbb{C}$;

(iii) An open set Ω is a norm-controlled calculus domain for σ if and only if $\Omega \supset \mathrm{hor}\,(\sigma)$.

Indeed, assertion *(iii)* is a straightforward consequence of *(i)* and theorem 4.2.2. Assertions *(i)* and *(ii)* are special cases of theorem 4.1.5, because $\delta_1^0(A, \hat{S}_b) = 1/2$ and $\varphi_e(f) = f(0) \in \hat{f}(\hat{S}_b)$ for every $f \in \mathcal{M}(S)$. ■

4.3.3. **Full spectral hulls equal to $\overline{\mathbb{D}}$.** Here we describe, following [N4], a class of “bad” Banach algebras for which the full spectral hull of *every* spectrum is equal to $\overline{\mathbb{D}}$. To this end, we need a bit of model operators. All properties claimed below can be found in [N1].

Let Θ be a singular inner function in $\mathbb{D}$, and let

$$\Theta = \Theta_\mu = exp\Big(\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, d\mu(\zeta)\Big), \quad z \in \mathbb{D},$$

be its canonical integral representation, where μ is a positive measure on $\mathbb{T}$ singular with respect to the Lebesgue measure m. Further, let $K_\Theta = H^2 \ominus \Theta H^2$ be the orthogonal complement of the corresponding z-invariant subspace ΘH^2 of the Hardy space H^2. The compression

$$f \longmapsto M_\Theta f = P_\Theta z f, \quad f \in K_\Theta,$$

of the multiplication operator is called a model operator; P_Θ stands for the orthogonal projection to K_Θ. The Sz.-Nagy-Foias model theory tells that every completely non unitary contraction T with $rank(1 - T^*T) = 1$ and $\sigma(T) \neq \overline{\mathbb{D}}$ is unitarily equivalent to an operator M_Θ, see [SzNF].

Now, we define the algebra $A = A_\Theta$ as *the norm closure of polynomials in M_Θ*. It is an exercise to show that always $\mathfrak{M}(A_\Theta) = \sigma(M_\Theta) = \mathrm{supp}\,(\mu) \subset \mathbb{T}$ and $\|M_\Theta\| = 1$. It is worth mentioning that, for $\Theta = \Theta_\mu$, where $\mu = \delta_1$, there exists a unitary operator $U : K_\Theta \longrightarrow L_2(0,1)$ such that the algebra $UA_\Theta U^{-1}$ is the norm closure of polynomials in the integration operator $Jf(x) = \int_0^x f(t)\, dt$, $x \in (0,1)$, on the space $L^2(0,1)$.

THEOREM 4.3.4. *[N4] Let $A = A_\Theta$ be the above Banach algebra corresponding to a singular inner function $\Theta = \Theta_\mu$ with the representing measure μ whose closed support is a* *set of zero Lebesgue measure,* $m(\mathrm{supp}\,(\mu)) = 0$*. Then for every closed subset* $\sigma \subset \overline{\mathbb{D}}$*,* $\sigma \neq \emptyset$*, we have* $h(\sigma; \mathfrak{M}(A)) = \overline{\mathbb{D}}$*.*

4.3.5. **Spectrally closed sets.** It is natural to call a subset $\sigma \subset \mathbb{C}$ (A, X)-*spectrally closed* (or A-*spectrally closed* in the case where $X = \mathfrak{M}(A)$) if $h(\sigma; A, X) = \sigma$. Property 4.1.2 tells us that every closed subset $\sigma \subset \overline{\mathbb{D}}$ is (A, X)-closed if and only if $\delta_1(A, X) = 0$. In Section 7 we give an account of known results about weighted Beurling-Sobolev algebras satisfying this property.

Part II. Convolution Algebras

5. Unweighted Convolution Algebras on Groups and Semigroups

Let G be an LCA group and $\mathcal{M}(G)$ be the convolution algebra of all complex Borel measures on G endowed with the standard variation norm $\|\mu\| = \mathrm{Var}\,(\mu)$. The Fourier transforms $\mathcal{F}\mu = \hat{\mu}$,

$$\mathcal{F}\mu(\gamma) = \hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\mu(x), \quad \gamma \in \hat{G},$$

form an algebra of functions on the dual group $\hat{G}$. We denote it by $\mathcal{F}M(G)$ and endow it with the range norm $\|\mathcal{F}\mu\| = \|\mu\|$. Clearly, $\mathcal{F}(\mu * \nu) = (\mathcal{F}\mu) \cdot (\mathcal{F}\nu)$ for all $\mu, \nu \in \mathcal{M}(G)$, and so the Banach algebras $\mathcal{M}(G)$ and $\mathcal{F}M(G)$ are isometrically isomorphic. The term “the measure algebra of G” will be referred to the both.

Since the mapping $\mu \longmapsto \hat{\mu}(\gamma)$ is a complex homeomorphism of $\mathcal{M}(G)$, we can injectively embed $\hat{G}$ into $\mathfrak{M}(\mathcal{M}(G))$, and regard $\hat{G}$ as the *visible spectrum* of $\mathcal{M}(G)$ in the sense of the Introduction. In particular, $\mathrm{clos}\,\hat{\mu}(\hat{G}) \subset \sigma(\mu)$ for any $\mu \in \mathcal{M}(G)$. The Wiener-Pitt-Sreider theorem implies (see the Introduction) that $\mathfrak{M}(\mathcal{M}(G)) = \mathrm{clos}\,\hat{G}$ if and only if G is a discrete group; in fact, in this case we simply have $\mathfrak{M}(\mathcal{M}(G)) = \hat{G}$.

However, even in the latter case, the problem of the norm-controlled inversion, as described in previous sections, is still of interest. Moreover, from the *quantitative point of view*, we cannot distinguish any advantage of discrete groups, for which the spectrum $\hat{G} = \mathfrak{M}(\mathcal{M}(G))$ is completely visible (in the sense of Definition 1.2.1), as compared with the general LCA groups, for which $X = \hat{G} \neq \mathfrak{M}(\mathcal{M}(G))$, and the spectrum is even 1-invisible. This is an essential distinction between the concepts of n-visibility (without any norm control) and ($\delta - n$)-visibility. Speaking informally, the 1-visibility of $\mathfrak{M}(\mathcal{M}(G))$ for discrete groups, guaranteed by the classical Wiener-Lévy theorem, is illusory because it does not endure quantitative specifications by ($\delta - 1$)-estimates of inverses.

5.1. *An upper estimate for the measure algebra on a group*

The constants $c_n(\delta, \mathcal{M}(G), \hat{G})$, defined in 1.2.3, can be estimated by using theorem 2.2.7, because the algebra $\mathcal{M}(G)$ is $\hat{G}$-symmetric in the sense of 2.2.1. The main condition *(iii)* of theorem 2.2.7 can be checked with the help of lemma 2.2.6. Namely, one can prove that $\varphi_e \in \mathrm{conv}\,(\delta_\gamma : \gamma \in \hat{G})$ by means of "triangular" positive semidefinite kernels k_α on G, such that $\hat{k}_\alpha(\gamma) \geqslant 0$ for $\gamma \in \hat{G}$, $0 \leqslant k_\alpha(x) \leqslant 1 = k_\alpha(0)$ for $x \in G$, and $k_\alpha(x) = 0$ outside of a neighborhood V_α of the origin such that $\bigcap_\alpha V_\alpha = \{0\}$. Indeed, setting

$$\varphi_\alpha(\mu) = \int_{\hat{G}} \hat{k}_\alpha(\gamma)\hat{\mu}(\gamma)\, dm_{\hat{G}}(\gamma),$$

we have $\varphi_\alpha \in \mathrm{conv}\,(\delta_\gamma : \gamma \in \hat{G})$ and $\lim_\alpha \varphi_\alpha(\mu) = \lim_\alpha \int_G k_\alpha\, d\mu = \mu(\{0\}) = \varphi_e(\mu)$ for all $\mu \in \mathcal{M}(G)$. This proves the following theorem.

THEOREM 5.1.1. *For every LCA group* G*, the spectrum of* $\mathcal{M}(G)$ *is completely* δ*-visible for every* δ *satisfying* $\frac{1}{\sqrt{2}} < \delta \leqslant 1$*, and, consequently,* $\delta_n(\mathcal{M}(G), \hat{G}) \leqslant \frac{1}{\sqrt{2}}$ *for all* $n \geqslant 1$*. Moreover,*

$$c_n(\delta, \mathcal{M}(G), \widehat{G}) \leqslant \frac{1}{2\delta^2 - 1} \tag{5.1}$$

for $1/\sqrt{2} < \delta \leqslant 1$ *and for all* $n \geqslant 1$*.*

5.2. *The measure algebra on a semigroup*

Here we consider the convolution measure algebras $\mathcal{M}(S)$ on semigroups S. Since the language of the semigroup theory is not canonically fixed, we define exactly which objects we are dealing with. For general facts of harmonic analysis on semigroups we refer to [T].

5.2.1. **Definitions.** By a *semigroup* S we mean the following.

(i) S is a Borel subset of a LCA group G such that $x, y \in S \Rightarrow x + y \in S$*,*

(ii) $0 \in S$.

A *bounded character* (also called semicharacter) on S is a bounded continuous function $\gamma : S \longrightarrow \mathbb{C}$ such that $\gamma(0) = 1$ and $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in S$. It is clear that every such function is bounded by 1: $|\gamma(x)| \leqslant 1$, $x \in S$. The *set of all bounded characters* of S is denoted by $\hat{S}_b$. Obviously, $\hat{G} \subset \hat{S}_b$, in the sense that the restriction $\gamma|S$ of a character $\gamma \in \hat{G}$ is a bounded character of S. In what follows, we assume that the following *separation property* holds.

(iii) For every $x \in S$, $x \neq 0$, *there exists a bounded character* $\gamma \in \hat{S}_b$ *such that* $|\gamma(x)| < 1$.

In particular, $S \cap (-S) = \{0\}$ if a semigroup S satisfies condition *(iii)*. Let $\mathcal{M}(S) = \{\mu \in \mathcal{M}(G) : \mu|(G\backslash S) \equiv 0\}$ be the subspace of $\mathcal{M}(G)$ consisting of all measures supported by S. An immediate verification shows that $\mathcal{M}(S)$ is a (closed) subalgebra of $\mathcal{M}(G)$. Now, we define the *Fourier-Laplace transformation* on $\hat{S}_b$ setting $\mathcal{L}\mu(\gamma) = \hat{\mu}(\gamma) = \int_S \gamma(x)\, d\mu(x)$ for $\mu \in \mathcal{M}(S)$ and $\gamma \in \hat{S}_b$. Clearly, the functional

(5.2) $$\mu \longmapsto \mathcal{L}\mu(\gamma), \quad \mu \in \mathcal{M}(S),$$

is a norm continuous homomorphism of the algebra $\mathcal{M}(S)$ for every $\gamma \in \hat{S}_b$. Moreover, $\mathcal{L}\mu(\gamma) = \mathcal{F}\mu(-\gamma)$ for all $\gamma \in \hat{G}$. Hence, it is natural to consider the space of bounded characters $\hat{S}_b$ as *the visible part* of $\mathfrak{M}(\mathcal{M}(S))$, and write $\hat{S}_b \subset \mathfrak{M}(\mathcal{M}(S))$.

The following theorem can be proved in a similar way to 5.1.1. However, the result is two times better as compared with the algebras $\mathcal{M}(G)$.

THEOREM 5.2.2. *Let S be a semigroup satisfying conditions (i)–(iii). Then,*

(5.3) $$\delta_1(\mathcal{M}(S), \widehat{S}_b) \leqslant \frac{1}{2},$$

and

(5.4) $$c_1(\delta, \mathcal{M}(S), \widehat{S}_b) \leqslant \frac{1}{2\delta - 1}$$

for all δ, $\frac{1}{2} < \delta \leqslant 1$.

5.3. *Examples and comments*

5.3.1. **Symmetric and analytic Wiener algebras.** Let $G = \mathbb{Z}$ be the additive group of integers, and let $S = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geqslant 0\}$. Then $\mathcal{M}(\mathbb{Z}) = l^1(\mathbb{Z})$, $\mathcal{M}(\mathbb{Z}_+) = l^1(\mathbb{Z}_+)$, and $W = FM(Z) = \{\hat{\mu} = \sum_{n\in Z} \mu(n)\zeta^n : \mu \in l^1(Z)\}$ is the Wiener algebra of absolutely converging Fourier series on the circle group $\hat{\mathbb{Z}} = \mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The bounded characters of $\mathbb{Z}_+$ fill in the closed unit disc $\hat{S}_b = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$. The corresponding Fourier-Laplace transformation is $\mu \longmapsto \hat{\mu}(z) = \sum_{n\geqslant 0} \mu(n)z^n$, and $W_+ = LM(Z_+) = \{\hat{\mu} = \sum_{n\geqslant 0} \mu(n)z^n : \mu \in l^1(Z_+)\}$ is the analytic Wiener algebra on $\overline{\mathbb{D}}$. Preceding theorems tell that $\|f^{-1}\|_W \leqslant (2\delta^2 - 1)^{-1}$ if $f \in W$ and $1/\sqrt{2} < \delta \leqslant |f(\zeta)| \leqslant \|f\|_W \leqslant 1$ for $|\zeta| = 1$, and $\|f^{-1}\| \leqslant (2\delta - 1)^{-1}$ if $f \in W_+$ and $1/\sqrt{2} < \delta \leqslant |f(z)| \leqslant \|f\|_{W_+} \leqslant 1$ for $|z| \leqslant 1$.

5.3.2. **Several variables, continuous versions, and cones in $\mathbb{Z}^N$ and $\mathbb{R}^N$.** The same estimates of inverses as in 5.3.1 hold for the multivariate Wiener algebra on the torus $\mathbb{T}^N$,

$$W = \mathcal{F}M(\mathbb{Z}^N) = \{\hat{\mu} = \sum_{n\in\mathbb{Z}^N} \mu(n)\zeta^n : \mu \in l^1(\mathbb{Z}^N)\},$$

for the analytic Wiener algebra on the polydisc $\overline{\mathbb{D}}^N$,

$$W_+ = \mathcal{L}M(\mathbb{Z}_+^N) = \{\hat{\mu} = \sum_{n\in\mathbb{Z}_+^N} \mu(n)z^n : \mu \in l^1(\mathbb{Z}_+^N)\},$$

and for continuous versions of the Wiener algebras. The latter correspond to $G = \mathbb{R}$, $S = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$, and—in several variables—to $G = \mathbb{R}^N$, $S = \mathbb{R}_+^N$. Here $\hat{G} = \mathbb{R}$, $\hat{S}_b = \overline{\mathbb{C}}_+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) \geqslant 0\}$ - the closed upper half-plane, and, for several variables, $\hat{G} = \mathbb{R}^N$, $\hat{S}_b = \overline{\mathbb{C}}_+^N$.

Instead of the cones $\mathbb{R}_+^N \subset \mathbb{R}^N$ and $\mathbb{Z}_+^N = \mathbb{R}_+^N \cap \mathbb{Z}^N$, we may consider an arbitrary subsemigroup S of $\mathbb{R}^N$ containing 0 and such that $S\backslash\{0\}$ is contained in an *open* halfspace (this requirement guarantees the separation property *(iii)* of Subsection 5.2.1). Usually, S is a convex cone satisfying the latter property, or, in its discrete version, the intersection of such a cone with $\mathbb{Z}^N$. For the continuous version, the semicharacters are $x \longmapsto e^{i(x\cdot z)}$ with $z \in \hat{S}_b$, where $\hat{S}_b = \{z \in \mathbb{C}^N : \mathrm{Im}\,(x \cdot z) \geqslant 0$ for all $x \in S\}$ and $x \cdot z = \sum_{k=1}^N x_k z_k$. The Fourier-Laplace transformation is again the classical one. For instance, taking $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geqslant 0,\ -kx_1 \leqslant x_2 \leqslant kx_1\}$, where $k > 0$, we have $\hat{S}_b = \{(z_1, z_2) \in \mathbb{C}^2 : k|\mathrm{Im}\,(z_2)| \leqslant \mathrm{Im}\,(z_1)\}$.

Another example is the halfspace $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\} \cup \{0\}$, with the corresponding dual set of characters $\hat{S}_b = \{(z_1, z_2) \in \mathbb{C}^2 : \Im(z_2) = 0,\ \mathrm{Im}\,(z_1) \geqslant 0\} = \mathbb{C}_+ \times \mathbb{R}$.

5.4. *Some subalgebras of $\mathcal{M}(G)$ and $\mathcal{M}(S)$*

Here we briefly consider two classical subalgebras of $\mathcal{M}(G)$ (or $\mathcal{M}(S)$), namely, the algebras of absolutely continuous, respectively, discrete measures on G (or on S).

5.4.1. **The group algebra $L^1(G)$.** Let G be an LCA group, and $L^1(G)$ the convolution group algebra on G, which becomes a unital algebra if we adjoin the Dirac point mass at the origin $e = \delta_0$ (if G is not discrete). Since the Riemann-Lebesgue lemma implies that $\varphi_e(\mu) = \lambda = \lim_{\gamma \longrightarrow \infty} \hat{\mu}(\gamma)$ for $\mu = \lambda e + f dm \in (L^1(G) + \mathbb{C}\cdot e)$, we can directly apply lemma 2.2.3 and obtain the following improvement of theorem 5.1.1: for any nondiscrete LCA group G, one has $\delta_1(L^1(G) + \mathbb{C}\cdot e, \hat{G}) \leqslant 1/2$; moreover, $c_1(\delta, L^1(G) + \mathbb{C}\cdot e, \hat{G}) \leqslant (2\delta - 1)^{-1}$ for all $1/2 < \delta \leqslant 1$. (For $n \geqslant 2$ we still have estimates (5.1)). In Subsection 5.5 we show that this new estimate for $L^1(G) + \mathbb{C}\cdot e$ is sharp.

5.4.2. **The algebra of almost periodic functions $\mathcal{F}M_d(G)$, and the algebra of Dirichlet series $\mathcal{L}M_d(S)$.** Let $\mathcal{M}_d(G)$ be the algebra of discrete measures on an LCA group G, and $\mathcal{M}_d(S)$ be its subalgebra of measures supported by a subsemigroup S satisfying hypotheses *(i)–(iii)* of Subsection 5.2.1.

The algebra $\mathcal{F}M_d(G)$ of Fourier transforms of discrete measures is the algebra of almost periodic functions with absolutely convergent Fourier series. Theorem 5.1.1 works for this algebra too (the visible spectrum is, of course, $X = \hat{G}$). As for the entire algebra $\mathcal{M}(G)$, we will see in Subsection 5.5 below that $\delta_1(\mathcal{F}M_d(G), \hat{G}) \geqslant 1/2$. It should be mentioned that, as before, impossibility of the norm control of inverses for small δ, i.e., the fact that $c_1(\delta, \mathcal{M}_d(G), \hat{G}) = \infty$ for $0 < \delta \leqslant 1/2$, is not related to the evident fact that the spectrum

$\mathfrak{M}(\mathcal{M}_d(G)) = (\hat{G})^-$ (the Bohr compactum) is much larger than $X = \hat{G}$. Indeed, we show that even staying on the Bohr compactum we still have $\delta_1(\mathcal{M}_d(G), (\hat{G})^-) \geqslant 1/2$, see below.

Similarly, $\mathcal{F}M_d(S)$ is the algebra of absolutely convergent Dirichlet series

$$f(\gamma) = \sum_{x \in S} \mu(\{x\})(x, \gamma), \gamma \in \hat{S}_b$$

with $\sum_{x \in S} |\mu(\{x\})| < \infty$. The case of the classical Dirichlet series corresponds to the case $S = \mathbb{R}_+$, $\hat{S}_b = \{z \in \mathbb{C} : Re(z) \geqslant 0\}$ with the pairing $(x, z) \longmapsto e^{-zx}$. Like $\mathcal{M}_d(G)$, the algebra $\mathcal{M}_d(S)$ is an inversion stable subalgebra of $\mathcal{M}(S)$. Hence, Theorem 5.2.2 is still valid for this algebra as well.

5.5. *A lower estimate for the measure algebras*

In this section, we show how to get a common lower estimate of $c_1(\delta, A, X)$ for the measure algebras of groups and semigroups, and for their subalgebras considered above. In particular, we show that the critical constant δ_1 is greater than or equal to $\frac{1}{2}$ for every infinite LCA group and for the most part of semigroups.

In fact, we dispose two approaches to lower estimates. The first one is based directly on the existence and properties of measures carried by independent Cantor sets; in what follows a special kind of such measures (Sreider measures) is used. The corresponding theory is surely one of the most subtle chapters of measure algebra theory, see [GRS], [Ru1], [GMG]. The second method is inspired by H. Shapiro's example [Sh1]. In our setting, it depends on some improvements of the technique of exponentials norm behaviour $\|e^{it\mu}\|_{\mathcal{M}(G)}$ for $t > 0$. This technique is well known to be the ground of the theory of functions operating on an algebra, see [Ru1], [GMG]. The first way is faster, and, following [N4], we use it in this section. For the second one, more explicit, we refer to [ENZ] and [Sh2].

We start by recalling some classical facts on measure algebras.

5.5.1. **Sreider measures.** Let G be a nondiscrete LCA group. It is known that there exists a positive measure $\mu \in \mathcal{M}(G)$ such that $\hat{\mu}(\hat{G}) \subset [-1, 1]$ and $\hat{\mu}(\mathfrak{M}) = \overline{\mathbb{D}}$, $\|\mu\| = 1$; see Sreider [Sr] for $G = \mathbb{R}$, and [Ru1], Theorem 5.3.4, for the general case and history. We call such μ's *Sreider measures*. Moreover, it is known that there exist continuous Sreider measures μ such that the convolution powers and their translates, μ^k, $\delta_x * \mu^k$, are all mutually singular, see [Ru1], [GMG]. It is worth mentioning that $0 \in \hat{\mu}(\hat{G})$ for all known examples of such measures.

5.5.2. **Bohr compactification.** For an LCA group G, we denote by $\hat{G}_d$ the dual group $\hat{G}$ endowed with the discrete topology, and set $\overline{G} = (\hat{G}_d)\hat{}$. The compact group $\overline{G}$ is called the Bohr compactification of G; in fact, G is a dense subset of $\overline{G}$, and $\overline{G} = \mathfrak{M}(\mathcal{M}(\hat{G}_d)) = \mathfrak{M}(\mathcal{M}_d(\hat{G}))$.

5.5.3. **How to get lower estimates on groups.** Here we explain a method to prove lower estimates for critical constants and norm-controlling constants, making use of Sreider measures and Bohr compactification. One can call the method "a walk to the Bohr compactum". In particular, we get the needed estimates for all three algebras on groups we considered above, namely, for $\mathcal{M}(G)$, $L^1(G) + \mathbb{C} \cdot \delta_0$, and $\mathcal{M}_d(G)$.

The method consists of the following steps (see [N4] for more details):

1) staying on an infinite group G, we lift ourselves up to the Bohr group $\overline{G} \supset G$;

2) being nondiscrete, $\overline{G}$ carries a Sreider measure, say ν, whose polynomials give the required lower estimate, see Lemma 5.5.4 below;

3) using almost periodic approximations of $\mathcal{F}\nu$, we obtain the same lower estimate on G (via the classical Bochner criterion for the membership in $\mathcal{F}M(G)$, see [Ru1], Theorem 1.9.1).

Of course, for a nondiscrete group G, steps 1) and 3) are not necessary. To accomplish step 3) for general G, we need a kind of a strengthened weak topology, precisely, a version of Beurling's narrow topology. Namely, we say that a net $(\mu_i)_{i\in I} \subset \mathcal{M}(G)$ is $\hat{G}$*-convergent* to a measure $\mu \in \mathcal{M}(G)$ if $\underline{\lim}_{i\in I}\|\mu_i\| \leqslant \|\mu\|$ and $\lim_{i\in I} \hat{\mu}_i(\gamma) = \hat{\mu}(\gamma)$ for $\gamma \in \hat{G}$. A set $A \subset \mathcal{M}(G)$ is said to be $\hat{G}$*-dense* in a set $B \subset \mathcal{M}(G)$ if every $\mu \in B$ is a $\hat{G}$-limit of a net of measures belonging to A.

The following lemma is a straightforward consequence of the spectral mapping theorem.

LEMMA 5.5.4. *Let ν be a Sreider measure on a nondiscrete LCA group G, and let $\mu = \delta e + (1-\delta)\nu^2$, where $1/2 < \delta \leqslant 1$, and $e = \delta_0$ stands for the unit of $\mathcal{M}(G)$. Then $\|\mu\| = 1$, $\hat{\mu}(\hat{G}) \subset [\delta, 1]$ and $\|\mu^{-1}\| = (2\delta - 1)^{-1}$.*

THEOREM 5.5.5. *Let G be an infinite LCA group, and let A be a unital subalgebra of $\mathcal{M}(G)$ (not necessarily closed) satisfying the following conditions: (i) A is $\hat{G}$-symmetric; (ii) A is $\hat{G}$-dense in $\mathcal{M}(G)$. Further, given a number δ, $1/2 < \delta \leqslant 1$, let $A([\delta, 1]) = \{\alpha \in A : \|\alpha\| \leqslant 1,\ \hat{\alpha}(\hat{G}) \subset [\delta, 1]\}$. Then*

$$\sup_{\alpha \in A([\delta,1])} \|\alpha^{-1}\| \geqslant (2\delta - 1)^{-1}.$$

In particular, $\delta_1(A, \hat{G}) \geqslant 1/2$, and $c_1(\delta, A, \hat{G}) \geqslant (2\delta - 1)^{-1}$ for $\frac{1}{2} < \delta \leqslant 1$.

The algebras $A = \mathcal{M}(G)$, $A = L^1(G) + \mathbb{C}\cdot e$, and $A = \mathcal{M}_d(G)$ satisfy conditions (i) and (ii), and hence the conclusions hold for these algebras.

5.5.6. **How to get a lower estimate on semigroups.** Here we describe a semigroup counterpart of theorem 5.5.5 by using the same method of moving to the Bohr compactum. However, the construction of measures giving the maximum to $\|\mu^{-1}\|$ when $\inf|\hat{\mu}|$ is fixed is necessarily different. Indeed, the previous construction is based on the $\hat{G}-$ symmetry of $\mathcal{M}(G)$ and on positive semidefiniteness of the corresponding Fourier transforms. Both reasons fail for semigroups. Instead, we are using a more complicated, but general construction of theorem 2.3.1. By the way, this gives another proof to theorem 5.5.5; namely, theorem 2.3.1 and lemma 5.5.7 below imply 5.5.5.

In order to realize the technique of narrow approximations on a semigroup instead of the entire group, we restrict ourselves to a class of semigroups S described in Subsection 5.2.1. This class contains all frequently used semigroups, in particular, all examples of Subsection 5.3 above.

Until the end of this Section, we denote by $\mathcal{A}$ a subalgebra of $\mathcal{M}(G)$, and by A a subalgebra of $\mathcal{M}(S)$.

LEMMA 5.5.7. *Let G be an infinite LCA group, and let $\mathcal{A}$ be a subalgebra of $\mathcal{M}(G)$ which is $\hat{G}-$symmetric and $\hat{G}-$dense in $\mathcal{M}(G)$ (but not necessarily closed and/or unital). Then, $\mathcal{A}$ satisfies (2.3) with $X = \hat{G}$.*

Now, we describe a class of semigroups S and a class of subalgebras $A \subset \mathcal{M}(S)$ where the method based on theorem 2.3.1 works; see [N4] for more details.

5.5.8. **Absorbing semigroups and S-subalgebras.** Let S be a semigroup embedded into an LCA group G, $S \subset G$, and satisfying hypotheses *(i)–(iii)* of Subsection 5.5.1. We say that G satisfies the *absorbtion condition* if the following is true.

(iv) For every compact set $K \subset G$ there exists an element $x \in G$ such that $x + K \subset S$.

If S is absorbing, there exists an element $x \in S$ such that $x + K \subset S$ (consider $K \cup \{0\}$), and, therefore, *(iv)* implies that S generates G in the sense $G = S - S$. On the other hand, if S is generating and $S \backslash \{0\}$ is open, or S contains an open generating part, then S is absorbing. For instance, this is the case for all examples of Subsection 5.3.

Now, we describe the class of subalgebras of $\mathcal{M}(S)$ we are working with. Namely, let S be a semigroup, $S \subset G$. We say that A is *an S-subalgebra of* $\mathcal{M}(S)$ if A is *a subalgebra of* $\mathcal{M}(S)$ containing a "small" subalgebra of the form $P_S\mathcal{A} + \mathbb{C}\cdot e$, where $\mathcal{A} \subset M(G)$ stands for a subalgebra of $\mathcal{M}(G)$ verifying the following conditions (compare with the conditions of theorem 5.5.5):

(i) $\mathcal{A}$ is $\hat{G}-$symmetric;

(ii) $\mathcal{A}$ is $\hat{G}-$dense in $\mathcal{M}(G)$;

(iii) $\mathcal{A}$ is S-invariant, that is, $\delta_x * \mathcal{A} \subset A$ for $x \in S$;

(iv) $\hat{G}$ is a boundary for $\mathfrak{M}(\mathcal{A})$, that is, $\lim_n \|\mu^n\|^{1/n} = \sup_{\hat{G}} |\hat{\mu}|$ for every $\mu \in \mathcal{A}$.

Observe that $\mathcal{A}$ is not assumed to be unital. The standard subalgebras $A = \mathcal{M}_d(S)$ and $A = L^1(S) + \mathbb{C}\cdot e$, with obvious group counterparts $\mathcal{A} = \mathcal{M}_d(G)$ and $\mathcal{A} = L^1(G)$, respectively, are S-subalgebras of $\mathcal{M}(S)$. Hence, the same is true of any bigger subalgebra. For instance, $\mathcal{M}_d(S) + L^1(S)$, and $\mathcal{M}(S)$ itself, are S-subalgebras. Another example is the algebra $\mathcal{M}_f(S)$ of finitely supported measures on S.

Now, we are ready to state the following lower estimate for semigroups.

THEOREM 5.5.9. *Let S be a semigroup satisfying conditions (i)–(iii) of Subsection 5.2.1 and the absorbtion condition (iv). Let A be an S-subalgebra of $\mathcal{M}(S)$ (not necessarily closed). Then $\delta_1(A, \hat{S}_b) \geqslant 1/2$ and $c_1(\delta, A, \hat{S}_b) \geqslant (2\delta - 1)^{-1}$ for all δ, $1/2 < \delta < 1$.*

For the proof, we check (2.3) with $X = \hat{S}_b$, see [N4], theorem 3.3.7.

6. Some Finite Groups and Semigroups

In this section we briefly consider the measure algebras $\mathcal{M}$ on finite groups and on finite semigroups. In general, exact computations of the majorants $c_n(\delta, \mathcal{M}, X)$ for these groups and semigroups are, probably, even more complicated than in the infinite case. This is why we mainly restrict ourselves to two examples: to cyclic groups $C_d = \mathbb{Z}/d\mathbb{Z}$ of order $d \geqslant 1$, to nilpotent semigroups $Z_d = \mathbb{Z}_+/(d + \mathbb{Z}_+)$ of order d. In a sense, the groups C_d "exhaust" the group $\mathbb{Z}$ and the semigroups Z_d "exhaust" $\mathbb{Z}_+$. Essentially, we consider the asymptotics of $c_n(\delta, \mathcal{M}(C_d), \hat{C}_d)$ and $c_n(\delta, \mathcal{M}(Z_d), \hat{Z}_d)$ as $d \longrightarrow \infty$.

6.1. *Finite groups*

6.1.1. **Preliminaries.** Let G be a finite group written additively, and m_G the invariant (Haar) measure normalized by $m_G(\{x\}) = 1$ for every $x \in G$. Clearly, the space $\mathcal{M}(G) = L^1(G) = l^1(G)$ is a convolution Banach algebra with the unit $e = \delta_0$. All complex homomorphisms are given by the Fourier (Gelfand) transformation, $f \longmapsto \mathcal{F}f(\gamma) = \hat{f}(\gamma) = \sum_{x \in G} f(x)(-x, \gamma)$, $\gamma \in \hat{G}$, where $\hat{G}$ is the dual group of unimodular characters written multiplicatively. Therefore, $\mathfrak{M}(\mathcal{M}(G)) = \hat{G}$. The Haar measure $m = m_{\hat{G}}$ is normalized to have total mass 1, so that the Fourier transformation $\mathcal{F}$ is a unitary operator from $L^2(G)$ to $L^2(\hat{G})$. It is easy to see that we can regard the dual group $\hat{G}$ as the visible spectrum for any convolution algebra on G. That is, in our previous notation, we set $X = \hat{G}$.

Using equivalence of every two norms on a finite dimensional vector space, one can show that for finite groups the majorants $c_1(\delta, A, \hat{G})$ always have the linear growth rate as $\delta \longrightarrow 0$. For example, $\|\mu^{-1}\| \leqslant k(A)/\delta$ for every $\mu \in A$ satisfying $|\hat{\mu}(\gamma)| \geqslant \delta$ for all $\gamma \in \hat{G}$. Here $k(A)$ is a constant depending only on a convolution algebra A on a finite group G, and $k(\mathcal{M}(G)) \leqslant (\mathrm{card}\,(G))^{1/2}$.

We can say more for the special case of cyclic groups C_d.

6.2. *The cyclic group C_d*

Let $G = C_d = \mathbb{Z}/d\mathbb{Z} = \{0, 1, \ldots, d-1\}$ be the cyclic group endowed with the quotient composition. The dual group $\hat{C}_d$ is the group of d-th roots of unity $\hat{C}_d = \{\zeta_k = \zeta^k : 0 \leqslant k \leqslant d-1\}$, where $\zeta = \zeta(d) = e^{2\pi i/d}$. The Fourier transform of an element $f \in \mathcal{M}(C_d)$ is $Ff(\zeta_k) = \hat{f}(\zeta_k) = \sum_{0 \leqslant s < d} f(s)\zeta_k^s$, $\zeta_k \in \hat{C}_d$, and the norm is $\|f\| = \sum_{0 \leqslant s < d} |f(s)|$. Let $e_k = \chi_{\{k\}}$ be the basic functions on C_d. The convolution on C_d, which we denote by $\circ$, follows the rule $e_r \circ e_s = e_t$, where $t \in C_d$, $t \equiv (r+s) \bmod (d)$. Since we cannot compute $c_n(\delta, \mathcal{M}(C_d), \hat{C}_d)$, we consider the behaviour of the upper bound $c_n(\delta, \{C_d\})$ defined by

(6.1) $$c_n(\delta, \{C_d\}) = \sup\{c_n(\delta, \mathcal{M}(C_d), \widehat{C}_d) : d \geqslant 1\}, \quad 0 < \delta \leqslant 1.$$

The following theorem shows that, in a sense, the algebras $\mathcal{M}(C_d)$ approximate the algebra $\mathcal{M}(\mathbb{Z}) = l^1(\mathbb{Z})$ as $d \longrightarrow \infty$.

THEOREM 6.2.1. *(i)* $c_n(\delta, \mathcal{M}(C_d), \hat{C}_d) \leqslant d^{1/2} \cdot \min(\delta^{-2}, \delta^{-1} n^{1/2})$ *for all* $0 < \delta \leqslant 1$ *and* $n \geqslant 1$.

(ii) $c_n(\delta, \mathcal{M}(C_d), \hat{C}_d) \leqslant (2\delta^2 - 1)^{-1}$ *for all* $1/\sqrt{2} < \delta \leqslant 1$ *and* $n \geqslant 1$.

(iii) $c_n(\delta, \{C_d\}) \geqslant c_n(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T})$ *for all* $0 < \delta \leqslant 1$*, where* $c_n(\delta, \{C_d\})$ *is defined in (6.1). In particular,* $c_1(\delta, \{C_d\}) = \infty$ *for* $0 < \delta \leqslant 1/2$*, and* $c_1(\delta, \{C_d\}) \geqslant (2\delta - 1)^{-1}$ *for* $1/2 < \delta \leqslant 1$.

6.3. *The nilpotent semigroup* Z_d

The semigroup $Z_d = \mathbb{Z}_+/(d+\mathbb{Z}_+)$ is defined as the set $Z_d = \{0, 1, \ldots, d-1, d\}$ endowed with the operation $(s,t) \longmapsto \min(s+t, d)$, and with the measure $m(\{s\}) = 1$ for $0 \leqslant s < d$ and $m(\{d\}) = 0$. Therefore, on the basic functions e_s, $e_s(t) = \delta_{s,t}$ (the Kronecker delta), the convolution is defined by the formula

$$e_s \circ e_t = e_{s+t} \quad \text{for} \quad s+t < d, \quad \text{and} \quad e_s \circ e_t = 0 \quad \text{for} \quad s+t \geqslant d.$$

The space $\mathcal{M}(Z_d) = L^1(Z_d, m)$ of all measures (functions) on Z_d endowed with this convolution and with the usual L^1 norm is a unital d-nilpotent Banach algebra. Namely, the algebra $\mathcal{M}(Z_d)$ has a generator e_1 such that $e_1^d = 0$. Hence, the only character on Z_d is the trivial one: $0 \longmapsto 1$ and $s \longmapsto 0$ for $s > 0$. We write $\hat{Z}_d = \{0\}$, and $\mathfrak{M}(\mathcal{M}(Z_d)) = \{0\}$ with the only homomorphism on $\mathcal{M}(Z_d)$, namely $\mu = (\mu(s))_{0 \leqslant s < d} \longmapsto \mu(0)$.

Theorem 6.3.1 below gives the exact value of the majorant $c_1(\delta, \mathcal{M}(Z_d), \{0\})$ and shows that the algebra $\mathcal{M}(\mathbb{Z}_+) = l^1(\mathbb{Z}_+)$ is, in a sense, the limit of $\mathcal{M}(Z_d)$ as $d \longrightarrow \infty$.

THEOREM 6.3.1. *For all* $0 < \delta \leqslant 1$, $c_1(\delta, M(Z_d), \{0\}) = \delta^{-1} \sum_{0 \leqslant k < d} (\frac{1-\delta}{\delta})^k$*, and, therefore,* $c_1(\delta, \mathcal{M}(Z_d), \{0\}) \sim \delta^{-d}$ *as* $\delta \longrightarrow 0$*. Moreover,*

$$c_1(\delta, \{Z_d\}) =: \sup\{c_1(\delta, \mathcal{M}(Z_d), \{0\}) : d \geqslant 1\} = c_1(\delta, \mathcal{M}(\mathbb{Z}_+), \overline{\mathbb{D}})$$

for all $0 < \delta \leqslant 1$*, that is,* $c_1(\delta, \{Z_d\}) = \infty$ *for* $0 < \delta \leqslant 1/2$*, and* $c_1(\delta, \{Z_d\}) = (2\delta - 1)^{-1}$ *for* $1/2 < \delta \leqslant 1$.

7. The Weighted Beurling-Sobolev Algebras

In this section, we present some results on estimates of the inverses in Beurling-Sobolev algebras contained in [ENZ]. The principal conclusion is that the spectrum $\mathfrak{M}(A)$ of a "sufficiently smooth" Beurling-Sobolev algebra $A = l^p(\mathbb{Z}, w)$, as well as of its analytic counterpart $A = l^p(\mathbb{Z}_+, w)$, is $(\delta - 1)$-visible for every $\delta > 0$, that is $\delta_1(A, \mathfrak{M}(A)) = 0$. We also show explicit upper bounds for $c_1(\delta, A, \mathfrak{M}(A))$ for $\delta > 0$. The weights w are subject to some regularity conditions.

In particular, the analytic Wiener algebra $l^1(\mathbb{Z}_+)$, which admits no norm control for the inverses for $0 < \delta \leqslant 1/2$ (see Section 5), turns out to be the only exception in the scale of weighted algebras $l^1(\mathbb{Z}_+, w)$, up to the weight regularity mentioned above. The reason for this difference lies in a sort of "asymptotic compactness" of the algebra multiplication in the presence of a non-trivial weight regularly tending to infinity, and in its absence for the unweighted case; see comments to theorem 7.2.4 for the definitions. Technically, the phenomenon mentioned is measured by the compactness of the embedding of the algebra $l^p(\mathbb{Z}_+, w)$ or $l^p(\mathbb{Z}, w)$ in the multiplier algebra $\mathrm{mult}\,(l^p(w'))$ of the space of derivatives $l^p(w') = \mathcal{D}l^p(w)$, as is required by the method of Section 3.

7.1. *The Beurling-Sobolev algebras* $l^p(w)$

We start by fixing notation and recalling some basic facts on the weighted convolution algebras $l^p(w)$, and on the Beurling-Sobolev algebras; by our definition, the latter represent a special case of $l^p(w)$. Then we describe regularly growing weights generating a Beurling-Sobolev algebra.

7.1.1. **The topological Banach algebras** $l^p(\mathbb{Z}, w)$**.** For a positive function (a weight) $w : \mathbb{Z} \longrightarrow (0, \infty)$ and an exponent $1 \leqslant p < \infty$, we denote

$$l^p(\mathbb{Z}, w) = \{x = (x_n)_{n\in\mathbb{Z}} : xw \in l^p(\mathbb{Z})\} = \{x : \|x\|_{p,w} < \infty\},$$

where $\|x\|_{p,w} = (\sum_{n\in\mathbb{Z}} |x_n|^p w(n)^p)^{1/p} < \infty$, with the usual modification for $p = \infty$, $\|x\|_{\infty,w} = \sup_{n\in\mathbb{Z}} |x_n w(n)| < \infty$. The convolution of two finitely supported sequences is defined in the usual way, $x * y = (\sum_{k\in\mathbb{Z}} x_k y_{n-k})_{n\in\mathbb{Z}}$. It is well known that the convolution can be extended to a continuous multiplication on $l^p(\mathbb{Z}, w)$ provided that

$$C_{p,w} = \sup_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \left(\frac{w(n)}{w(k)w(n-k)} \right)^{p'} \right)^{1/p'} < \infty, \tag{7.1}$$

where p' stands for the conjugate exponent, $1/p + 1/p' = 1$. The case $p = 1$ is classical. For $p > 1$, the sufficiency of the continuous analogue of (7.1) for the weighted space $L^p(\mathbb{R}, w)$ to be a convolution algebra was proved by J. Wermer in [W]. For the case of $l^p(\mathbb{Z}, w)$, the sufficiency of (7.1), as well as the necessity of it in the case $p = \infty$, was proved in [N3] (without knowing about Wermer's result). In [KL], it was shown that, in general, (7.1) is not necessary, but for an even ($w(-n) = w(n)$) and $\log$-concave weight w this condition is necessary for any p. However, a practically useful necessary and sufficient condition for regularly varying weights first appeared only in [ENZ], see below theorem 7.1.5. For information on weighted convolution L^p algebras on $\mathbb{R}^n$, see [KS].

Assuming (7.1), we get $\|x * y\|_{p,w} \leqslant C_{p,w} \|x\|_{p,w} \|y\|_{p,w}$ for every $x, y \in l^p(\mathbb{Z}, w)$. Hence, $l^p(\mathbb{Z}, w)$ becomes a unital topological Banach algebra with the unit $e = e_0 = (\delta_{0k})_{k\in\mathbb{Z}}$. For

$p = 1$, the condition $w(0) = 1$, $w(n+k) \leqslant w(n)w(k)$ obviously is necessary and sufficient for $l^1(\mathbb{Z}, w)$ to be a Banach algebra. It is an exercise to show that for $p > 1$ the norm $\|\cdot\|_{p,w}$ is *never* a Banach algebra norm; on the contrary, for $p = 1$ every topological Banach algebra weight is equivalent to a Banach algebra weight.

By a *Beurling-Sobolev algebra* we mean the space $l^p(\mathbb{Z}, w)$ satisfying condition (7.1) and endowed with the norm $\|\cdot\|_{p,w}$. We recall that the l^1 weighted Banach algebras are usually called *Beurling algebras*; in the general case of the algebras $l^p(\mathbb{Z}, w)$, there are reasons for adding the name of S. L. Sobolev, because for $p = 2$ and $w(n) = (1 + |n|)^\alpha$ we get the standard Sobolev spaces (algebras) on $\mathbb{T}$ as the spaces $\mathcal{F}l^p(\mathbb{Z}, w)$ of Fourier transforms.

7.1.2. **Maximal ideals. A necessary condition.** Since $A = l^p(\mathbb{Z}, w)$ satisfies axioms *(i)* and *(ii)* of Subsection 3.1, the spectrum of $l^p(\mathbb{Z}, w)$ is the annulus $A(r_-, r_+)$, $A(r_-, r_+) = \{\lambda \in \mathbb{C} : r_-(w) \leqslant |\lambda| \leqslant r_+(w)\}$, where $r_- = r_-(w) = \lim_{k \longrightarrow -\infty} w(k)^{1/k} > 0$, $r_+ = r_+(w) = \lim_{k \longrightarrow +\infty} w(k)^{1/k} < \infty$.

The Gelfand transform is given by $x \longmapsto \hat{x}$,

$$\hat{x}(\lambda) = \mathcal{F}x(\lambda) = \sum_{k \in \mathbb{Z}} x_k \lambda^k, \lambda \in A(r_-, r_+).$$

Moreover, it follows from the inclusion $\mathcal{F}l^p(\mathbb{Z}, w) \subset C(A(r_-, r_+))$ that the algebra $l^p(\mathbb{Z}, w)$ is always embedded into the weighted algebra

$$l^1(Z, r_\pm^n) = \{x : \sum_{k<0} |x_k| r_-^k + \sum_{k \geqslant 0} |x_k| r_+^k < \infty\}.$$

This provides a *necessary condition* for $l^p(\mathbb{Z}, w)$ to be an algebra, namely, $(r_\pm^n / w(n)) \in l^{p'}(\mathbb{Z})$, or, equivalently,

$$\sum_{k \geqslant 0} \Big(\frac{r_+^k}{w(k)}\Big)^{p'} < \infty, \quad \sum_{k<0} \Big(\frac{r_-^k}{w(k)}\Big)^{p'} < \infty,$$

with the usual modification for $p' = \infty$.

7.1.3. **Remarks and notation.** With appropriate modifications, similar facts are true for the *analytic Beurling-Sobolev algebras* $l^p(\mathbb{Z}_+, w)$. In this Section, we consider the case of *positive spectral radius only*, $r_+(w) > 0$.

In the case of $l^p(\mathbb{Z}, w)$, we often assume a sort of weak symmetry of w, requiring that $r_- = r_+$. Without loss of generality, we can normalize the weight so as to have $r_+ = 1$. A general method for obtaining algebras $l^p(\mathbb{Z}, w)$ with arbitrary spectral radii is as follows. Let $l^p(\mathbb{Z}, w)$ be a Beurling-Sobolev algebra, and $R = (R_-, R_+)$ be two positive numbers such that $R_- \leqslant R_+$. Then, if we set $W_R(n) = R^n_{\text{sign}\,(n)} w(n)$ for $n \in \mathbb{Z}$, we get an algebra weight satisfying $C_{p,W_R} \leqslant C_{p,w}$ and $r_\pm(W_R) = R_\pm r_\pm$. In particular, starting with $r_- = r_+ = 1$, we can obtain an arbitrary pair of spectral radii.

Notation. We use the symbol l^p, respectively $l^p(w)$, as common notation for $l^p(\mathbb{Z})$ and $l^p(\mathbb{Z}_+)$, respectively for $l^p(\mathbb{Z}, w)$ and $l^p(\mathbb{Z}_+, w)$.

7.1.4. **The Beurling-Sobolev algebras $l^p(w)$.** Here we give regularity conditions on w guaranteeing that $l^p(w)$ is a Beurling-Sobolev algebra.

THEOREM 7.1.5. *I. Let $1 \leqslant p \leqslant \infty$, and $v = \log(w)$ be a positive eventually concave function on $\mathbb{R}_+$ such that $r_+ = \lim_{x \longrightarrow \infty} w(x)^{1/x} = 1$. Assume that the following condition is fulfilled: either, (a) there exists $\alpha > 0$ such that $w(x)/x^\alpha$ eventually decreases; or, (b) there exists $\alpha > 1/p'$ such that $w(x)/x^\alpha$ eventually increases and has concave logarithm.*

Then the following assertions are equivalent.

(i) The space $l^p(\mathbb{Z}_+, w)$ is an algebra.

(ii) The space $l^p(\mathbb{Z}_+, w)$ is a Beurling-Sobolev algebra.

(iii) $1/w \in L^{p'}(\mathbb{R}_+)$, or, equivalently, $(1/w(n))_{n \geqslant 0} \in l^{p'}$.

II. Let w be a weight function on $\mathbb{Z}$ such that $r_- = r_+ = 1$. Assume that $(w(n))_{n\geqslant 0}$ and $(w(-n))_{n\geqslant 0}$ are quasiincreasing sequences, that is $\sup_{n,j\geqslant 0} w(n)w(n+j) < \infty$, and similarly for $(w(-n))_{n\geqslant 0}$. Then $l^p(\mathbb{Z}, w)$ is a Beurling-Sobolev algebra if and only if so are $l^p(\mathbb{Z}_+, w)$ and $l^p(\mathbb{Z}_-, w)$.

7.1.6. **Examples.** *(i) $l^p(|n|_*^\alpha)$ is a Beurling-Sobolev algebra if and only if $\alpha p' > 1$ for $p > 1$, and $\alpha \geqslant 0$ for $p = 1$. Here $x_* = \max(x, 1)$ for $x \in \mathbb{R}$.*

(ii) If $l^1(w_\alpha)$ is a Beurling algebra, $w_\alpha(n) = w(n)/|n|_*^\alpha$, and $\alpha p' > 1$, then $l^p(w)$ is a Beurling-Sobolev algebra.

7.2. *When is the embedding $l^p(w) \subset \mathrm{mult}\,(\mathcal{D}l^p(w))$ compact?*

As was shown in Section 3, the compactness of this embedding plays a crucial role in the estimates of the norms of inverses. The best polynomial approximations, measured by the ϵ_n characteristics of this embedding, make it possible to control the norms of the inverses. This completes the program proposed in Section 3 for estimates of $c_1(\delta, A, \mathfrak{M}(A))$ in the case of Beurling-Sobolev algebras $A = l^p(w)$. We consider the special case $p = 1$ of Beurling algebras separately, because in this case we can state an explicit necessary and sufficient condition for compactness. Similar analysis can be made for $l^\infty(w)$. For the Beurling-Sobolev algebras $l^p(w)$, $1 < p < \infty$, some broad sufficient conditions are obtained. In this Subsection, we follow the paper [ENZ].

The case of the group $\mathbb{Z}$ is slightly different from the semigroup $\mathbb{Z}_+$, because in these two cases the convolutions differ in nature. Namely, on $\mathbb{Z}$, to form $(x * y)_n$ we add products $x_k y_j$ with k and j both unbounded, and for $\mathbb{Z}_+$ this is not the case. In particular, this elementary remark explains the difference between the nature of the weight $\sigma(n) = \|e_n\|_{\mathrm{mult}}$ on $\mathbb{Z}_+$ from that on $\mathbb{Z}$, see 7.2.5–7.2.6. For instance, for any analytic Beurling-Sobolev algebra $A = l^p(\mathbb{Z}_+, w)$ we always have $A \subset \mathrm{mult}\,(A')$, but for the algebras $A = l^p(\mathbb{Z}, w)$, in general, this inclusion fails.

As before, the *weight* w' is defined by the formula $w'(n) = w(n)/|n|_*$, where $|n|_* = \max(|n|, 1)$. It is clear that $A' = \mathcal{D}A = l^p(w')$ for $A = l^p(w)$.

7.2.1. **Continuous embeddings** $l^p(w) \subset \mathrm{mult}\,(l^p(w'))$**.** As was just mentioned, an *analytic* Beurling-Sobolev algebra $l^p(\mathbb{Z}_+, w)$ is always contained in $\mathrm{mult}\,(l^p(\mathbb{Z}_+, w'))$.

THEOREM 7.2.2. *If* $l^p(\mathbb{Z}_+, w)$ *is a Beurling-Sobolev algebra, then*

$$l^p(\mathbb{Z}_+, w) \subset \mathrm{mult}\,(l^p(\mathbb{Z}_+, w'))$$

and the norm of this embedding does not exceed $C_{p,w}$.

It is easy to see that for the group case, that is, for $l^p(\mathbb{Z}, w)$ in place of $l^p(\mathbb{Z}_+, w)$, some more growth restrictions on w are required for the inclusion $A = l^p(\mathbb{Z}, w) \subset \mathrm{mult}\,(A') = \mathrm{mult}\,(l^p(\mathbb{Z}, w'))$. Indeed, the condition $c = \sup_k(\|e_k\|_{\mathrm{mult}\,(A')}/\|e_k\|_{p,w}) < \infty$ is necessary for such an inclusion; it implies that $w(k)w(-k) \geqslant \mathrm{const} \cdot |k|_*$ for all $k \in \mathbb{Z}$.

Thus, for every $1 \leqslant p < 2$ there exist regularly growing weights w satisfying the condition (iii) of theorem 7.1.5 but $c = \infty$ (take $w(k) = |k|_*^\alpha$ with $1/p' < \alpha < 1/2$). Consequently, there are algebras $l^p(\mathbb{Z}, w)$ for which the embedding $l^p(\mathbb{Z}, w) \subset \mathrm{mult}\,(l^p(\mathbb{Z}, w'))$ fails. On the contrary, for $p \geqslant 2$, any algebra $l^p(\mathbb{Z}, w)$ with regularly growing weight seems to be contained in $\mathrm{mult}\,(l^p(\mathbb{Z}, w'))$.

On the other hand, we can guarantee the embedding in question (and even the compactness of this embedding), if we require a stronger algebra condition. Below, we present some results from [ENZ] obtained by using two different approaches.

First, we give a simple sufficient condition for the embedding $l_0^p(w) \subset \mathrm{mult}\,(l^p(w'))$ factorizing it through an auxiliary l^1-space.

THEOREM 7.2.3. *I. For any weighted space* $A = l^p(w)$, *or* $A = l_0^p(w)$, *we have*

$$\|e_n\|_{\mathrm{mult}\,(A')} = \sigma(n),$$

where

$$\sigma(n) = \sup_k \frac{|k|_* w(k+n)}{|k+n|_* w(k)}, \tag{7.2}$$

$k, n \in \mathbb{Z}$, *or* $k, n \in \mathbb{Z}_+$, *respectively. Hence, we have the contractive embedding* $l^1(\sigma) \subset \mathrm{mult}\,(A')$.

II. The embedding $l^p(w) \subset l^1(\sigma)$ *is equivalent to*

$$\frac{\sigma}{w} \in l^{p'}. \tag{7.3}$$

Being valid, this embedding is automatically compact for $p > 1$, *and it is compact for* $p = 1$ *if and only if*

$$\lim_k \frac{\sigma(k)}{w(k)} = 0, \tag{7.4}$$

where $k \longrightarrow \infty$ in the case of $\mathbb{Z}_+$, and $|k| \longrightarrow \infty$ in the case of $\mathbb{Z}$.

III. For every weighted space $l^p(w)$, condition (7.3) for $p > 1$, and condition (7.4) for $p = 1$, imply that $l^p(w) \subset_c \mathrm{mult}\,(l^p(w'))$.

IV. For a Beurling algebra $A = l^1(\mathbb{Z}_+, w)$ the following are equivalent

a) the embedding $A \subset B = \mathrm{mult}\,(A')$ is compact,

b) $l^1(\mathbb{Z}_+, w) \subset l^1(\mathbb{Z}_+, \sigma)$ is compact,

c) $\lim_{n,k\longrightarrow\infty} \frac{w(n+k)}{w(n)w(k)} = 0$,

d) the multiplication in $l^1(\mathbb{Z}_+, w)$ is asymptotically compact.

Moreover, if $A_0 = \{x \in A : \|x\|_A \leqslant 1\}$ is the unit ball of $A = l^1(\mathbb{Z}_+, w)$, then $\epsilon_n(A_0, B) = \sup_{k>n}(\sigma(k)/w(k))$ where $\epsilon_n(A_0, B)$ stands for the best polynomial approximation of degree n as defined in 3.2.3.

"Asymptotically compact multiplication" mentioned above means that for every $\epsilon > 0$ there exists an integer N such that $\|x * y\| < \epsilon\|x\| \cdot \|y\|$ for every $x, y \in l^p(\mathbb{Z}_+, w)$ satisfying $x_k = y_k = 0$ for $0 \leqslant k < N$.

The conditions *a), b)* and *d)* are still equivalent (after obvious modifications) for Beurling algebras $l^1(\mathbb{Z}, w)$ on $\mathbb{Z}$, but the behaviour of the weight σ is quite different in the cases of $\mathbb{Z}$ and $\mathbb{Z}_+$. Examples of weights w, for which the rate of the best polynomial approximations $\epsilon_n(A_0, B)$ can be computed explicitly, can be provided using the known Hardy field of functions, see Bourbaki [Bou]. In particular *such a weight satisfying* $\lim_{x\longrightarrow\infty} x^{-1}\log(w(x)) = 0$ *generates an algebra $l^1(\mathbb{Z}_+, w)$ and satisfies the following dichotomy:*

(i) either $\lim_{x\longrightarrow\infty}(w(x)/x) = 0$, and then $\sigma(n, w)/w(n) \simeq w(n)^{-1}$;

(ii) or $\lim_{x\longrightarrow\infty}(w(x)/x) > 0$, and then $\sigma(n, w)/w(n) \simeq n^{-1}$.

7.2.4. **The best polynomial approximations related to the embedding**

$$l^p(w) \subset \mathrm{mult}\,(l^p(w'))$$

It is the key point of our approach. Knowing $\epsilon_m(A_0, B)$, it remains only to apply the theory of Section 3. As before, the cases of $\mathbb{Z}_+$ and $\mathbb{Z}$ are slightly different.

THEOREM 7.2.5. *Let $1 \leqslant p \leqslant \infty$; suppose that $v = \log(w)$ satisfies the conditions (a), (b), and (iii) of theorem 7.1.5. Then the space $l^p(\mathbb{Z}_+, w)$ is an algebra, the embedding $A = l^p(\mathbb{Z}_+, w) \subset B = \mathrm{mult}\,(l^p(\mathbb{Z}_+, w'))$ is compact, and the following estimates are valid for the best approximations $\epsilon_m(A_0, B)$ of the unit ball $A_0 \subset A$.*

If condition (a) is satisfied with an exponent $\alpha \leqslant 1$, then

$$\epsilon_m(A_0, B) \leqslant (\sum_{j>m} w(j)^{-p'})^{1/p'}.$$

If condition (b) is satisfied, then $\epsilon_m(A_0, B) \leqslant c/m$ for $\alpha > 1 + 1/p'$; $\epsilon_m(A_0, B) \leqslant c \cdot (\log(m))^{1/p'}/m$ for $\alpha = 1 + 1/p'$, and $\epsilon_m(A_0, B) \leqslant c/m^{\alpha - 1/p'}$ for $1/p' < \alpha < 1 + 1/p'$, where c stands for a constant depending on α and p.

As is mentioned above, for the case of $\mathbb{Z}$ we have some extra constraints in order that the embeddings in question would be compact. For instance, the condition

$$\lim_{|n|\longrightarrow 0}(\|e_n\|_B/\|e_n\|_A) = 0$$

is obviously necessary for $A = l^p(\mathbb{Z}, w) \subset_c B = \mathrm{mult}\,(l^p(\mathbb{Z}, w'))$. Since $\|e_n\|_B = \sigma(n) \geqslant |n|_* w(0)/w(-n)$, we obtain that if $\underline{\lim}_{|n|\longrightarrow\infty}|n|w(n)w(-n) > 0$, the embedding $l^p(\mathbb{Z}, w) \subset \mathrm{mult}\,(l^p(\mathbb{Z}, w'))$ cannot be compact.

THEOREM 7.2.6. *Let w be a weakly symmetric normalized weight, that is $r_+ = r_- = 1$. Each of the following conditions implies that $l^p(\mathbb{Z}, w)$ is a Beurling-Sobolev algebra compactly embedded into the multiplier space, $A = l^p(\mathbb{Z}, w) \subset_c B = \mathrm{mult}\,(l^p(\mathbb{Z}, w'))$, with the following upper bounds for the best polynomial approximations $\epsilon_m(A_0, B)$ of the unit ball $A_0 \subset A$. Here $\mathcal{E}_m(p, \alpha)$ stands for the right hand side of the corresponding inequality in theorem 7.2.5 if $\alpha > 1$, and $\mathcal{E}_m(p, \alpha) = c \cdot m^{1-2\alpha+1/p'}$ otherwise.*

If $C_{1,w_\alpha} < \infty$ for an exponent $\alpha > 2^{-1}(1 + 1/p')$, where $w_\alpha = (w(k)/|k|_*^\alpha)_{k\geqslant 0}$, then $\epsilon_m(A_0, B) \leqslant C_{1,w_\alpha}\mathcal{E}_m(p, \alpha)$.

If $C_{p,w_\alpha} < \infty$ and $\alpha > 1/2$, then $\epsilon_m(A_0, B) \leqslant C_{p,w_\alpha}\mathcal{E}_m(1, \alpha)$.

For slowly increasing weights, we refer to [ENZ] for a simple direct estimate for the best approximations $\epsilon_m(A_0, B)$.

7.3. *Some explicit estimates of the norm controlling constants* $c_1(\delta, l^p(w))$

Now we are ready to obtain explicit estimates of the inverses in the Beurling-Sobolev algebras. To this end, we combine theorems of Section 3 with the estimates of the rate of polynomial approximations $\epsilon_m(A_0, B)$ provided in Subsections 7.1–7.2. Recall that the majorant M of theorem 3.2.5 depends on the distribution function λ_0 of the sequence $\epsilon_m(A_0, B)$ and on the multiplier norms $\sigma(k)$ of the basis vectors of $l^p(w)$. In 7.3.1 below we supposing $r_+ = r_- = 1$; for the case where $r_- < r_+$, see the remark at the end of this Subsection.

THEOREM 7.3.1. *Let A be a Beurling-Sobolev algebra, $A = l_0^p(\mathbb{Z}, w)$ or $l_0^p(\mathbb{Z}_+, w)$, compactly embedded into $B = \mathrm{mult}\,(l^p(w'))$. Then $\delta_1(A, \mathfrak{M}(A)) = 0$, and for all $\delta > 0$ we have the estimate $c_1(\delta, A, \mathfrak{M}(A)) \leqslant w(0)\delta^{-1} + M_1(\delta)$, where M is given in theorem 3.2.5 and $\|e_j\|_B = \sigma(j)$ (see formula (7.2)), and the constants $\mathcal{E}$ and C depend on the constant (7.1) and on the norm of the embedding $l_0^p(\mathbb{Z}_+, w) \subset B = \mathrm{mult}\,(l^p(w'))$.*

7.3.2. **Examples of estimates of inverses on $\mathbb{Z}_+$.** Here are some typical Beurling-Sobolev algebras $A = l^p(\mathbb{Z}_+, w)$, $1 \leqslant p \leqslant \infty$.

(i) $w(n) = n_*^\alpha$, $\alpha > 1 + 1/p'$. Then, $c_1(\delta, l^p(\mathbb{Z}_+, n_*^\alpha)) \leqslant c(\alpha, p)/\delta^{4\alpha+2}$.

(ii) $w(n) = n_*^\alpha$, $\alpha = 1 + 1/p'$. Then, $c_1(\delta, l^p(\mathbb{Z}_+, n_*^\alpha)) \leqslant c(\alpha, p)(\log(1/\delta))^{\alpha/p'}/\delta^{4\alpha+2}$.

(iii) $w(n) = n_*^\alpha$, $1/p' < \alpha < 1 + 1/p'$. Then, $c_1(\delta, l^p(\mathbb{Z}_+, n_*^\alpha)) \leqslant c(\alpha, p)/\delta^{2\beta+1}$, where $\beta = (\alpha p' - 1)^{-1}$.

(iv) $w(n) = e^{n^\alpha}$, $0 < \alpha < 1$. Then, $c_1(\delta, l^p(\mathbb{Z}_+, w)) \leqslant c(\alpha, p) \cdot \exp\{d_\alpha \delta^{-2\alpha(1-\alpha)}\} \cdot \delta^{-2} \cdot \log \frac{1}{\delta}$.

7.3.3. **Examples of estimates of inverses on $\mathbb{Z}$.** Comparison of the cases of $\mathbb{Z}$ and $\mathbb{Z}_+$ given in Subsections 7.1 and 7.2 shows that the explicit estimates of the inverses for rapidly growing weights should be the same on $\mathbb{Z}$ and on $\mathbb{Z}_+$, up to the sharp values of constants. Here, "rapidly" means at least as fast as the linearly growing weight $w(n) = |n|_*$, $n \in \mathbb{Z}$. For slower weights, e.g., for $w(n) = |n|_*^\alpha$, $\alpha < 1$, our method gives faster growth of the constants $c_1(\delta, l_p(w))$ on $\mathbb{Z}$ than on $\mathbb{Z}_+$. This method stops completely at the exponent $\alpha = \frac{1}{2}(1 + \frac{1}{p'})$. For this case we refer to the recent paper [E], where a different approach based on a Björk's paper [B] is employed. We restrict ourselves to a few examples illustrating the above theory.

(i) $w(n) = |n|_*^\alpha$, $\alpha > 1 + 1/p'$ or $1 \leqslant \alpha \leqslant 1 + 1/p'$. Then one has the same results as in 7.3.2 *(i)*, *(ii)* and *(iii)*, with modified constants.

(ii) $w(n) = |n|_*^\alpha$, $2^{-1}(1+1/p') < \alpha < 1$. Then $c_1(\delta, l^p(\mathbb{Z}, |n|_*^\alpha)) \leqslant c(\alpha, p)/\delta^{2+2(1+\beta)(2-\alpha)}$.

Observe that the right hand side of the last inequality diverges to infinity for $\alpha \longrightarrow 2^{-1}(1 + 1/p') = \alpha_p$, because $\beta \longrightarrow \infty$. However, in order to get a finite majorant, we can consider a weight growing slightly faster than the critical weight $w(n) = |n|_*^{\alpha_p}$. This can be done using $w(n) = |n|_*^{\alpha_p}(\log(1 + |n|_*))^\gamma$.

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